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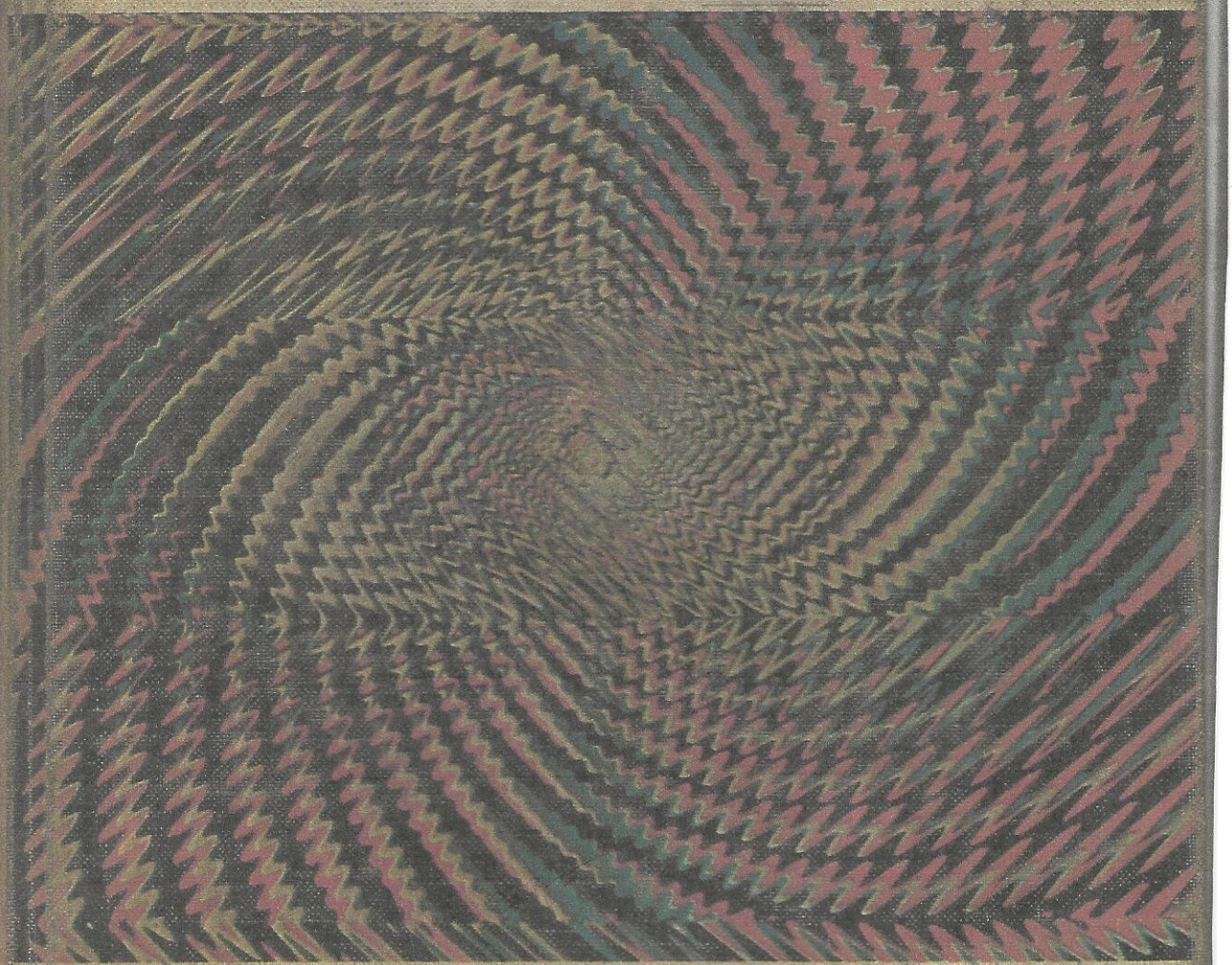

Joseph N. Payne Floyd F. Zamboni Francis G. Lankford, Jr.

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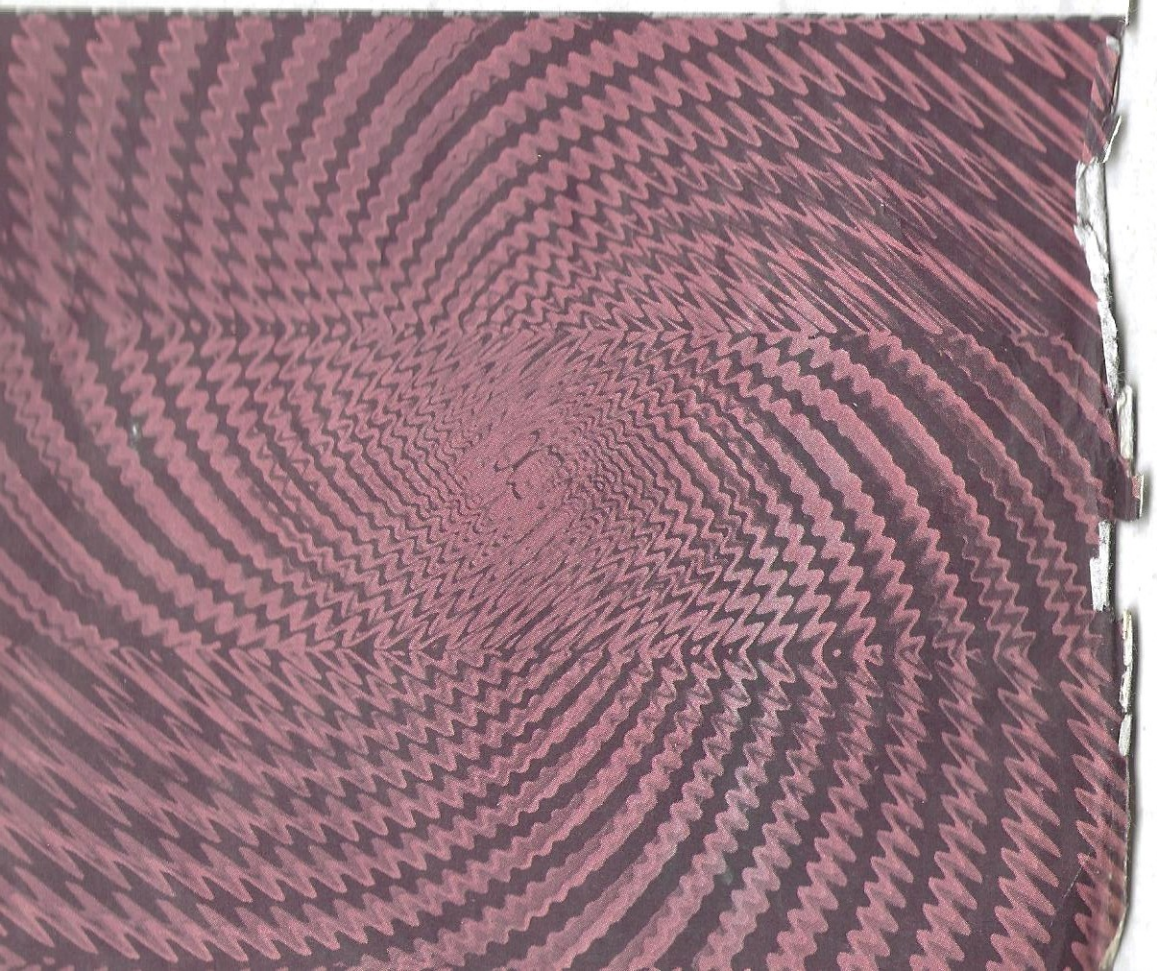

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ALGEBRA ONE

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ABOUT THE COVER

The cover shows a photograph taken by Sheldon S. Rose. He created the moiré pattern using four films of the same basic design, each in a different color. He placed the films one on top of the other and then photographed the result through a textured sheet of glass.

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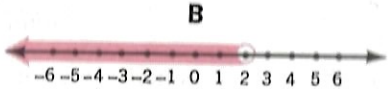
Find the solution set of each equation. Then write the letter for the correct graph of each inequality after the inequality. The set of real numbers is the replacement set in each case.

2. $7x = 14$
$7x > 14$
$7x < 14$

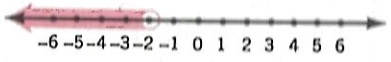
A

B



3. $-4x = 8$
$-4x > 8$
$-4x < 8$

A



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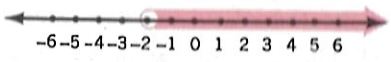
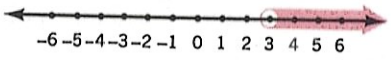


4. $8 - 5x = -7$
$8 - 5x > -7$
$8 - 5x < -7$

A



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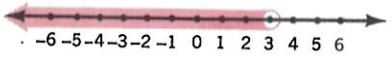
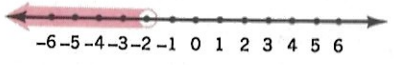


5. $17 - 4x = 7 - 9x$
$17 - 4x > 7 - 9x$
$17 - 4x < 7 - 9x$

A



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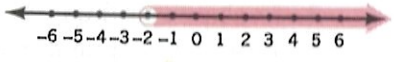


Solve and graph each inequality if the replacement set is the set of integers.

6. $x + 7 > 16$
7. $2x < 8$
8. $-4x > 12$
9. $2x + 2\frac{1}{2} > 10\frac{1}{2}$
10. $-6x < 0$
11. $16 < 3x + 4$
12. $7y < 20 - 3y$
13. $2t + 3 > 4t - 7$
14. $5 + 2x < 17$
15. $3x + 9 > -2$
16. $-3x - 4x > -21$
17. $5 - 6x < 17$
18. $4(5x + 7) < 13$
19. $-3x + 6(x - 4) > 9$

20. $2(x + 2) - 3(x - 1) < 0$ **21.** $4x + 12 > 7x + 3$

22. a. Solve $2x - 7 > 15$ by the method used in Section 3–7.

b. Now look at another way to solve the inequality.

	$2x - 7 > 15$
Add $+7$.	$2x - 7 + 7 > 15 + 7$
	$2x > 22$
Multiply by $\frac{1}{2}$.	$\frac{1}{2} \cdot 2x > \frac{1}{2} \cdot 22$
	$x > 11$

Is $\{x : x > 11\}$ the result you obtained in **a**?

c. In **b**, two postulates for inequalities are illustrated.

Postulate 3–4 Adding any nonzero number to both sides of an inequality produces an equivalent inequality.

Postulate 3–5 Multiplying both sides of an inequality by the same positive number produces an equivalent inequality.

Use the postulates to solve $3x + 9 < 19 + x$.

d. Find the solution set for $-x > 3$, using the same method as in Exercises 1–21. Does multiplying both sides by -1 give an equivalent inequality? Explain your answer.

e. In **d**, you should have seen that multiplying both sides by a negative number, -1, did not produce an equivalent inequality. But reversing the inequality sign *and* multiplying by -1 would produce an equivalent inequality.

	$-x > 3$
Multiply by -1 and reverse the inequality sign.	$(-1)(-x) < (-1)(3)$
	$x < -3$

Solution set: $\{x : x < -3\}$

This illustrates another postulate for inequalities.

Postulate 3–6 Multiplying both sides of an inequality by a negative number produces an equivalent inequality if and only if the inequality sign is reversed.

Polynomials

8

$\sqrt{3}$ $\quad 3x$

$-8x^2$

$-8x^2+3x+8$

$x^3-8x^2+3x+\sqrt{3}$

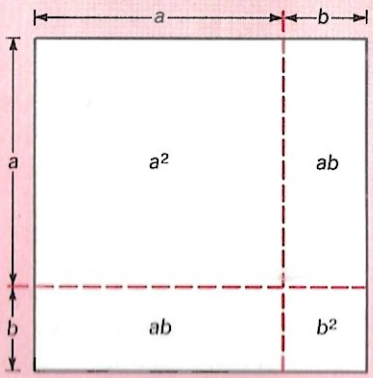
Nonpolynomials

$4\sqrt{3x}$ $\quad -\sqrt{3x}$

$-3x^{\frac{3}{2}}$ $\quad \frac{5}{x}$

$\frac{x}{x^2-3}$

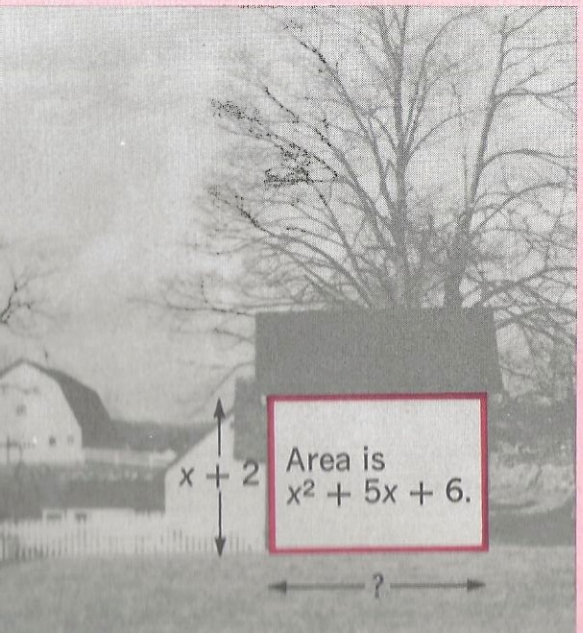
$\frac{x+3}{x^2-6}$

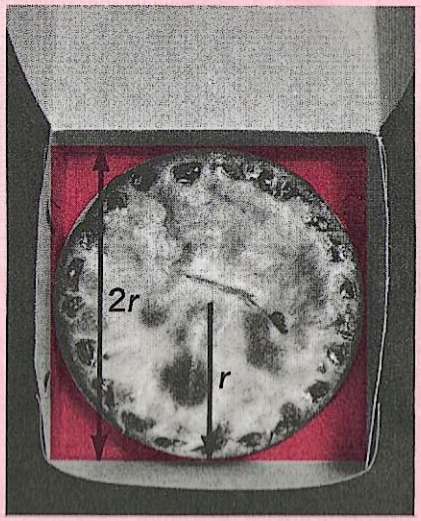


Area is $(a+b)^2$, or $a^2+2ab+b^2$.

$$(a+b)^2=a^2+2ab+b^2$$

$$101^2=(100+1)^2=100^2+2(100\cdot 1)^2+1^2$$
$$=10{,}000+\ ?\ +?$$
$$=\ ?$$





Area shaded $= (2r)^2-\pi r^2$
$= 4r^2-\pi r^2$
$= (4-\pi)(?)$

CHAPTER 7

POLYNOMIALS AND FACTORING

The diagrams show some uses of a subset of algebraic expressions called *polynomials*.

- Look at the examples in the upper left region to discover what a polynomial is. For polynomials, is any exponent of the variable negative? Is any exponent nonintegral? Does any polynomial involve division by the variable? How are polynomials different from nonpolynomials?
- In the upper right diagram, a and b represent positive numbers. What is the length of each side of the square? Why is the area of the square region $(a+b)^2$?
- Look at the center region. If the sentence $(a+b)^2=a^2+2ab+b^2$ is true for any real numbers a and b, how is this sentence used to square 101 easily?
- In the lower left photograph, one side of the rectangle measures $x+2$ units and the area of the rectangular region is x^2+5x+6 units. What is the length of the other side? In other words, is there a polynomial that multiplied by $(x+2)$ gives x^2+5x+6?
- What is the area of the square in the lower right photograph? What is the area of the circle? What is the area of the shaded region? Can you name the area of the shaded region using two factors, one of which is $(4-\pi)$?

In this chapter you will learn to answer questions such as these about products and factors of polynomials. You will also learn to solve some equations involving polynomial expressions.

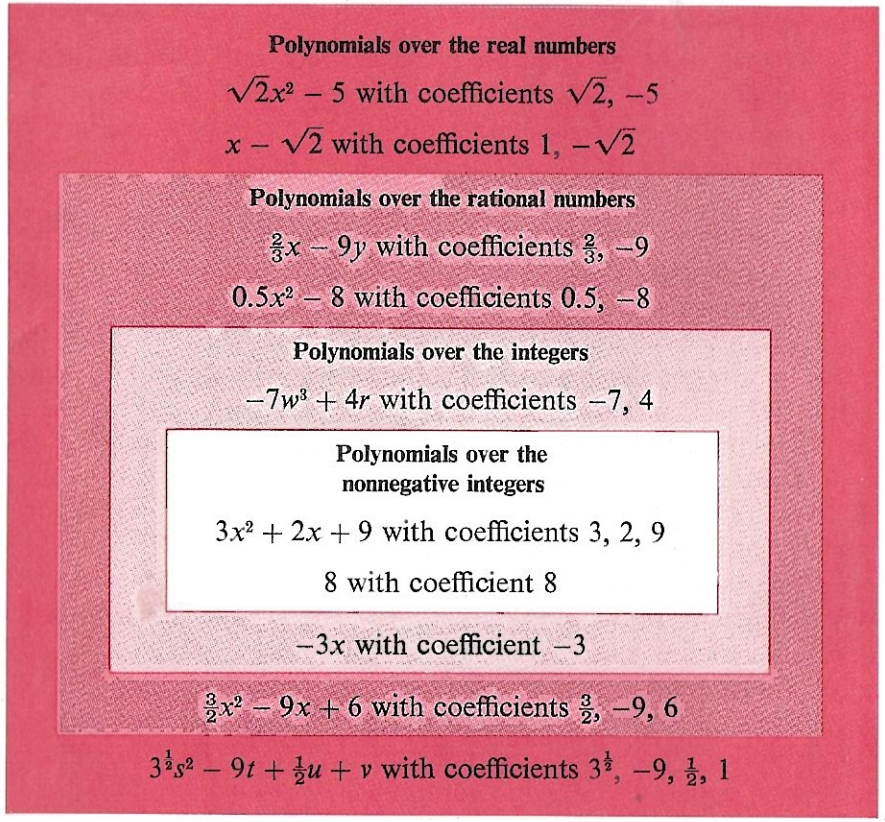
7–1 Classifying Polynomials

By studying the chapter opening, you should arrive at the following definition.

Definition A polynomial in one variable is an algebraic expression formed by adding or multiplying numbers and the variable a finite number of times, so that there are only nonnegative integral exponents for the variable.

The replacement set for the variable in all polynomials in this chapter is the set of real numbers. The polynomial $2x^2 - 5$, for example, names a real number when any real number is chosen for x.

Polynomials may be in more than one variable, and the coefficients of polynomials may be any real numbers. Polynomials may be classified according to the specified subset of real numbers used as coefficients, as in the diagram.


Polynomials over the real numbers
$\sqrt{2}x^2 - 5$ with coefficients $\sqrt{2}$, -5
$x - \sqrt{2}$ with coefficients 1, $-\sqrt{2}$
Polynomials over the rational numbers
$\frac{2}{3}x - 9y$ with coefficients $\frac{2}{3}$, -9
$0.5x^2 - 8$ with coefficients 0.5, -8
Polynomials over the integers
$-7w^3 + 4r$ with coefficients -7, 4
Polynomials over the nonnegative integers
$3x^2 + 2x + 9$ with coefficients 3, 2, 9
8 with coefficient 8
$-3x$ with coefficient -3
$\frac{3}{2}x^2 - 9x + 6$ with coefficients $\frac{3}{2}$, -9, 6
$3\frac{1}{2}s^2 - 9t + \frac{1}{2}u + v$ with coefficients $3\frac{1}{2}$, -9, $\frac{1}{2}$, 1

How does the diagram show that *all* the expressions are polynomials over the real numbers? Which expressions are polynomials over the rational numbers? over the integers?

Another way to classify polynomials is by the number of **terms**, or addends. Use the chart of prefixes and examples to help you remember the following names.

mono-	one	monotone
bi-	two	bicycle
tri-	three	tricycle

A **monomial** is a polynomial of one term: $2x^2$.
A **binomial** is a polynomial of two terms: $4a + 3$.
A **trinomial** is a polynomial of three terms: $16y^4 - 7y^3 - 4y^2$.

Classify each polynomial in the diagram as a monomial, a binomial, or a trinomial.

Exercises

A — Write *Polynomial* or *Nonpolynomial* to classify each of the following expressions.

1. $3x^3 + 2x - 8$
2. $3x^{\frac{1}{2}} - 2x + 8$
3. $\sqrt{3}x^2 - 4x + 9$
4. $5m^{-2} + n^{-1} + 7$
5. $\frac{3}{x} - 5x^2 + 6$
6. $15x$
7. 39
8. $5\sqrt[3]{a^2 + 5b}$
9. $\frac{a}{2} + 4$
10. $\frac{7}{2x - 5}$

— For each of the following polynomials, write *Nonnegative integers*, *Integers*, *Rational numbers*, or *Real numbers* to name the coefficients. You may have more than one answer.

11. $5z^2 - 4z + 11$
12. $1.5x^3 - x + 12$
13. $4x + 5$
14. $\pi x^2 - 2$
15. $\frac{x}{3} - 28$
16. $\sqrt{25}w^4 - 3x^2 + 12y$

17. Give the least inclusive set of numbers for the coefficients in Exercises 11–16.

— Write *Monomial*, *Binomial*, or *Trinomial* to classify each polynomial.

18. $3x^3 + 3$
19. $2s^2 - 2t + 2$
20. a^5
21. $19 - 7x^2 + 5x$
22. $\sqrt[3]{11}r^2 + 11s$
23. $\frac{x^2}{3}$

B — If x is -1, tell whether or not the following pairs of polynomials name the same number.

24. $x^3 + x^2 - x$ and $\frac{x^2}{4} + \frac{x}{2} + (\frac{1}{2})^2$

25. $\sqrt{2}x^2 + 2x - 3$ and $\sqrt{8}x^3 + 3\sqrt{2}x^2 - 5$

26. $0.2x^4 - 0.5x^3 + 1$ and $1.2x^3 - 1.5x^2 + 1$

27. $\frac{2x^3}{3} + \frac{x}{6} + \frac{1}{2}$ and $3\sqrt{3}x^3 + \sqrt{27}x^2 - x$

7–2 Addition and Subtraction of Polynomials

When you add $2a$ and $3a$ to get $5a$, you are adding monomials. The monomials can be added, or combined, only if the variables are the same, that is, if they are *like terms*.

To add polynomials, you proceed in the same way — combining like terms. The terms are usually arranged either in alphabetical order or in ascending and descending powers of the variable.

$2b - 3d + 4a$ would be $4a + 2b - 3d$,

in alphabetical order.

$\frac{1}{2}y^2 - 3xy + 7x^2 - 4y^3$ would be $7x^2 - 3xy + \frac{1}{2}y^2 - 4y^3$;

the powers of x descend; the powers of y ascend; and it is in alphabetical order.

You will make fewer mistakes if you think of the sign as the sign for the coefficient of the term and think of all terms as being added.

EXAMPLE 1. Add $4a + 6b$ and $2a - 3b$.

Arrange the polynomials under one another and add.

$$\begin{array}{r} 4a + 6b \\ 2a - 3b \\ \hline 6a + 3b \end{array}$$

EXAMPLE 2. Add $-3x^2 + 7xy - 6y^2$ and $5xy + 3y^2 - 4x$.

Notice that the second polynomial has a term that the first does not.

$$\begin{array}{rl} -3x^2 + 0 + 7xy - 6y^2 & \text{(Zero added for missing term.)} \\ \quad -4x + 5xy + 3y^2 & \text{(Rearranged in ascending powers of } y\text{.)} \\ \hline -3x^2 - 4x + 12xy - 3y^2 & \end{array}$$

To subtract polynomials, you proceed as in any subtraction example and add the additive inverse. The additive inverse of $6x^2$ is $-6x^2$. The additive inverse of $2n^2 + 3n^3m^2 - \frac{1}{2}m$ is $-2n^2 - 3n^3m^2 + \frac{1}{2}m$.

EXAMPLE 3. Subtract $a - b + c$ from $2a + b - c$.

Arrange the polynomials under one another.

$$\begin{array}{r} 2a + b - c \\ a - b + c \end{array}$$

Subtract.

$$\begin{array}{r} 2a + b - c \\ -a + b - c \\ \hline a + 2b - 2c \end{array}$$

Now try these

— Write the sum or difference in polynomial form.

1. $(4a - 3b) + (8a + 5b)$

2. $(4a^2 - 5ab - 6b^2) + (10ab - 6a^2 - 8b^2)$

3. $(6r + 9s) - (4r + 3s)$

4. $(h^2 + 6h + 9) - (25 - 10h + h^2)$

Answers: **1.** $12a + 2b$ **2.** $-2a^2 + 5ab - 14b^2$ **3.** $2r + 6s$ **4.** $16h - 16$

Exercises

A — Add.

1. $\begin{array}{r} 4a - 3b + 5c \\ 8a + 5b - 9c \end{array}$ **2.** $\begin{array}{r} 4r + 3s + t \\ 6r + 9s + 5t \end{array}$ **3.** $\begin{array}{r} 6p - 4t + x \\ -7p + 8t + 6x \\ -p - 4t + 7x \end{array}$

4. $5b + 4c + 8d$, $8b - 7c - 6d$, $3b - 4c$

5. $7x - 2y + 5z$, $8y - 9z$, $7x - 5y$

6. $5x + 7y - 10$, $8y - 2x + 3$, $2x - 10y + 4$

7. $4a^2 - 5ab - 6b^2$, $10ab - 6a^2 - 8b^2$, $10b^2 - 3a^2 - 7ab$

8. $a - b + c$, $b - c - a$, $c - a + b$, $a - 2b - c$

9. $4ab - 5bc + 6ac$, $6bc - 7ab - 8ac$, $10ab - bc - ac$

10. $x^2 + y^2 - z^2$, $3z^2 - 2x^2 - 4y^2$, $4y^2 + x^2 + z^2$

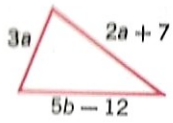
11. $x^2 - 6x + 7$, $8x - 15 - x^2$, $4x^2 + 6x - 9$, $7x + 10$

12. $a^2 - 2ab + b^2$, $2ab + a^2 + b^2$, $4a^2 - 4ab - b^2$, $a^2 - b^2$

13. $a + 2b$, $3b - 4c$, $5a - 7c$, $3b + 2d$

Write an algebraic expression, in simplest terms, for the perimeter of each of the following polygons.

14.



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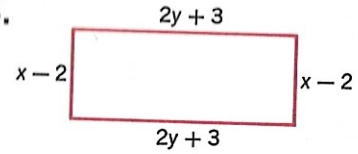

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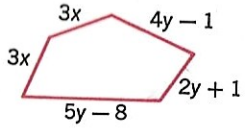

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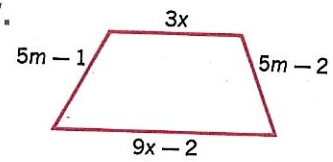

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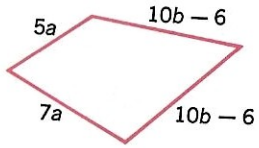

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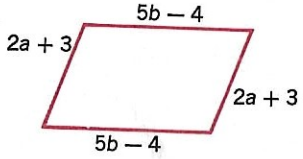


Subtract.

20. $3a + 4$ / $2a + 9$

21. $7b - 5$ / $10b - 8$

22. $r + 8s$ / $r - 5s$

23. $7x - 3$ / $-2x - 9$

24. $3x^2 + 8y^2$ / $x^2 - 8y^2$

25. $5m^2 - 9n$ / $8m^2 - 10n$

26. $8a - 9xy$ / $-a + 9xy$

27. $a - b + c$ / $a + b - c$

28. $x^2 + 2x + 1$ / $x^2 - 2x + 1$

29. $2a + 0$ / $a - b$

30. $2x^2 - 4$ / $x + 2$

31. $x^3 + 0 + 0 + 8$ / $x^3 + x^2 - 2x + 0$

32. $x^2 + 6$ / $x^3 - 3$

33. $3x^2 - 6x$ / $2x^2 + 7$

34. $-4x^2 - x + 9$ / $3x^3 + x$

35. From $3a - 2b + 5c$, take $-2a + 5b - c$.

36. Subtract $6x^2 - 3xy + y^2$ from $8x^2 + 5xy - y^2$.

37. From $6ab - 2ac + 5bc$, take $10ab - 2bc + 3ac$.

38. Subtract $x^2 - 2xy + y^2$ from $x^2 + 2xy + y^2$.

39. From $x^2 - 2xy + y^2$, take $x^2 + 2xy + y^2$.

40. From the sum of $x + 3y$ and $-3x - y$, subtract $x - y$.

41. Subtract $x^2 - y^2 - z^2$ from the sum of $2x^2 + 3y^2 - z^2$ and $4x^2 - 3y^2 + 5z^2$.

42. Take $a - b + 1$ from the sum of $a + c + 1$ and $a + b + 1$.

43. Subtract the sum of $(4x^3 + 2x^2 - 5)$ and $(15x - x^3 - x^2)$ from the difference of $(9 + 6x - 2x^2)$ and $(4x - 1 - 5x^3 + x^2)$.

44. If x is 3 inches, which is greater: the perimeter of a square whose sides each measure $3x - 1$ or the perimeter of a rectangle whose length is $x + 9$ and whose width is $2x - 3$? How much greater?

Combine like terms.

45. $2(a + 5) + 3(a + 5) - 4(a + 5) + 6(a + 5)$

46. $3a + 4 + 2(3a + 4) - 3(3a + 4)$

47. $8a + b - 5a + b - 2a + b$

48. When you add two polynomials over the set of nonnegative integers, will you obtain a polynomial over the same set? Show that your answer to the question is correct.

49. Repeat Exercise 48 for polynomials over the set of integers.

50. Repeat Exercise 48 for polynomials over the rational numbers.

51. Repeat Exercise 48 for polynomials over the real numbers.

52. What conclusion can you draw from the results in Exercises 48–51?

7–3 Products of Polynomials

The examples below show how products of polynomials may be found.

EXAMPLE 1. Find the product of $8x^2$ and $3x - 5$.

For the first step you may use the definition of subtraction, thinking of "$3x$ minus 5" as "$3x$ plus (-5)."

$$8x^2(3x - 5) = 8x^2[(3x) + (-5)]$$

Use distributive postulate.

$$= 8x^2(3x) + 8x^2(-5)$$

Use associative and commutative postulates and Theorem 6–1.

$$= 24x^3 - 40x^2$$

Is the product of $8x^2$ and $3x - 5$ also a polynomial over the integers?

EXAMPLE 2. Multiply $x + 2$ and $x - 5$.

You start by thinking of $x + 2$ as one number in applying the distributive postulate. Think of $x + 2$ as a in $a(b + c) = ab + ac$.

$$\begin{array}{rcl} a(b+c) & = & ab + ac \\ (x+2)(x-5) & = & (x+2)(x)+(x+2)(-5) \end{array}$$

Use distributive postulate.	$= x^2 + 2x - 5x - 10$
Use distributive postulate.	$= x^2 + (2 - 5)x - 10$
Use definition of subtraction.	$= x^2 - 3x - 10$

EXAMPLE 3. Find the product of $5t + 2n$ and $7t + 3n$.

$$\begin{aligned} (5t + 2n)(7t + 3n) &= (5t + 2n)(7t) + (5t + 2n)(3n) \\ &= 35t^2 + 14nt + 15nt + 6n^2 \\ &= 35t^2 + (14 + 15)nt + 6n^2 \\ &= 35t^2 + 29nt + 6n^2 \end{aligned}$$

If you prefer, you can write one polynomial under the other and multiply either from left to right or from right to left. The important thing is to keep like terms together.

$$\begin{array}{r} 7t + 3n \\ 5t + 2n \\ \hline 35t^2 + 15nt \\ 14nt + 6n^2 \\ \hline 35t^2 + 29nt + 6n^2 \end{array} \quad \begin{array}{l} \leftarrow 5t(7t + 3n) \\ \leftarrow 2n(7t + 3n) \end{array} \quad \begin{array}{r} 7t + 3n \\ 5t + 2n \\ \hline 14nt + 6n^2 \\ 35t^2 + 15nt \\ \hline 35t^2 + 29nt + 6n^2 \end{array}$$

Since $35t^2 + 29nt + 6n^2$ is another name for $(5t + 2n)(7t + 3n)$, the two expressions should name the same number for every pair (t, n). You can substitute a pair of numbers to see if they produce the same results in both expressions. Let $t = 2$ and $n = 3$.

$$\begin{aligned} &35t^2 + 29nt + 6n^2 \\ &= (35 \cdot 4) + (29 \cdot 6) + (6 \cdot 9) \\ &= 140 + 174 + 54 \\ &= 368 \end{aligned} \qquad \begin{aligned} &(5t + 2n)(7t + 3n) \\ &= (10 + 6)(14 + 9) \\ &= 16 \cdot 23 \\ &= 368 \end{aligned}$$

Although the results agree, this does not *prove* the answer is right. Why? The real reason for substituting is to remind you that the two expressions are names for the same number for *all* real numbers that replace t and n. In other words, the sentence

$$(5t + 2n)(7t + 3n) = 35t^2 + 29nt + 6n^2$$

is an identity.

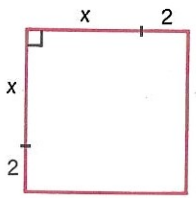
Exercises

A Rename each product in polynomial form. Check Exercises 1–6 by letting a be 2 and b be 3.

1. $(a + 3)(a - 4)$
2. $(2a + 3b)(a - b)$
3. $(2a - 5)(3a + 2)$
4. $(a + 3)(a + 2)$
5. $(2a + 3)(3a - 2)$
6. $(a - 3b)(2a + 3b)$
7. $(a + b)(c + d)$
8. $(a - b)(c + d)$
9. $(a - b)(c - d)$
10. $(a + b)(c - d)$
11. $(3b - 4c)(3b + 4c)$
12. $(3b - 4c)(2c + 3d)$
13. $(4x - 5)(4x + 5)$
14. $(7n - 3)(7n + 3)$
15. $(3x - 4)(3x - 4)$
16. $(2a - 3b)(a - 4b)$
17. $(5x - 3y)(3x + 4y)$
18. $(a + b)(a - b)$
19. $(\frac{1}{2}x - y)(2x + y)$
20. $(\frac{1}{3}t - 5)(\frac{1}{3}t + 6)$
21. $(0.2u + 4)(5u - 0.3)$
22. $(1.6n - 9)(0.2n - 5)$
23. $(a^2 + b)(a^2 - 2b)$
24. $(x^2 - y^2)(x^2 + y^2)$
25. $(m^2 - 2n^2)(m^2 - 2n^2)$
26. $(4s - 3)(2s^2 + 3)$
27. $(a^3 + b^3)(3a^3 - b^3)$
28. $(2x^2 + 3y^2)(3x^2 + 2y^2)$
29. $(x - 3)^2$
30. $(2x + 3y)^2$

Find the area of each region outlined in red.

31.



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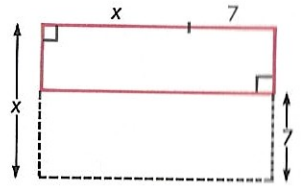

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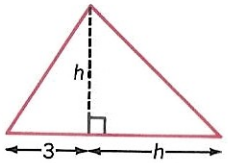

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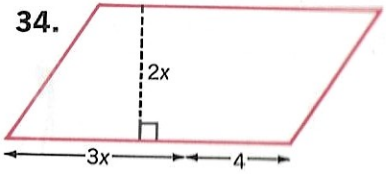

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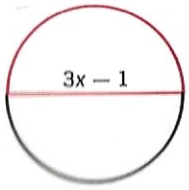

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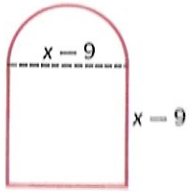


37. Note the difference in meaning in the expressions in **a** and **b**.

Expression	Meaning	Polynomial form
a. $(2x - 3)(3x + 4)$	$(2x - 3)$ times $(3x + 4)$	$6x^2 - x - 12$
b. $2x - 3(3x + 4)$	-3 times $(3x + 4)$ added to $2x$	$-7x - 12$

Check the work in **a** and **b** by substituting 5 for x in the expressions under **Expression** and **Polynomial form.**

— Rewrite each expression in polynomial form.

38. $(x + 5)(x - 7)$
39. $x + 5(x - 7)$
40. $y - 6(2y + 8)$
41. $(y - 6)(2y + 8)$
42. $2x + y(8x - 4y)$
43. $(2x + y)(8x - 4y)$
44. $7x - 9(3x - 4)$
45. $4 + (5x + 2)$
46. $2(8x + 3) - 5$
47. $5 - 2(8x + 3)$

B **48.** Look at the polynomials over the integers used as factors in Exercises 1–18. Is the product in every case also a polynomial over the integers? Is the set of polynomials over the integers closed with respect to multiplication?

49. Look at the polynomials over the rational numbers in Exercises 1–30. Is the set of polynomials over the rational numbers closed with respect to multiplication?

50. Write each product in polynomial form.

a. $(x - 9)(2x + 3)$ **b.** $(2x + 3)(x - 9)$

Do you think multiplication of polynomials is commutative? Test some other examples.

C — Write each product in polynomial form.

Example. $(n^2 - 3n + 2)(n - 2)$

$$\begin{array}{r} n^2 - 3n + 2 \\ n - 2 \\ \hline n(n^2 - 3n + 2) \longrightarrow n^3 - 3n^2 + 2n \\ -2(n^2 - 3n + 2) \longrightarrow \quad -2n^2 + 6n - 4 \\ \hline n^3 - 5n^2 + 8n - 4 \end{array}$$

51. $(x^2 - 2x + 1)(x - 1)$
52. $(x^2 + 2x + 1)(x + 1)$
53. $(x^2 - 2x + 1)(x + 1)$
54. $(x^2 + 2x + 1)(x - 1)$
55. $(x^2 - 6xy + 9y)(x - 3y)$
56. $(x^2 - 6x + 8)(x + 2)$
57. $(x^2 - 7xy + 12y^2)(x - 3y)$
58. $(n^3 - 5n^2 + 2)(n^2 - 2n)$
59. $(x^2 - xy + y^2)(x + y)$
60. $(x^2 + xy + y^2)(x - y)$

61. $(2 - 3a^2 - a)(a - 1)$
62. $(c - 2c^2 + 3)(2 + c)$
63. $(x^3 + x + 1)(x^2 - x + 1)$
64. $(x^3 + 3x^2 + 2x - 1)(x - 1)$
65. $(x^3 - 3x^2 + 3x - 1)(x - 1)$
66. $(5y^2 - 2 - 3y)(3 - y)$
67. $(3a^2 + 2 - 5a)(4a - 3)$
68. $(a^3 - 5 + 2a - 3a^2)(2 + a)$

69. If Q and R are polynomials over the rational numbers,

$$Q = ax^2 + bx + c \qquad \text{and} \qquad R = dx + e,$$

prove that QR is a polynomial over the rational numbers. (*Hint:* What kind of numbers are a, b, c, d, and e? Find the product, and use the postulates for rational numbers on page 31.)

7–4 Products of Binomials by Inspection

The product of two binomials, such as $2x - 7$ and $3x + 9$, is found so often in algebra that it is helpful to be able to write the product just from inspection of the two binomials.

Look at the examples below. When you first find the product, you have four terms, but the four can be simplified to three when the binomials have the same variables.

EXAMPLE 1. Find the product of $2x - 7$ and $3x + 9$ by inspection.

Previously your work would have looked like the following.

$$\begin{aligned} (2x - 7)(3x + 9) &= (2x - 7)(3x) + (2x - 7)(9) \\ &= 6x^2 - 21x + 18x - 63 \\ &= 6x^2 + (-21 + 18)x - 63 \\ &= 6x^2 - 3x - 63 \end{aligned}$$

Note that $-21x$ and $18x$ combine to give $-3x$. Which terms are multiplied to give $6x^2$? -63? $-21x$? $18x$?

To find the product by inspection, you need only

① multiply the first terms of each factor for the first term of the product,

② find the sum of the product of the inner terms (-7 and $3x$) and the product of the outer terms ($2x$ and 9) for the second term of the product,

③ multiply the last terms of each factor for the last term of the product.

EXAMPLE 2. Find $(4x + 1)(2x + 3)$ by inspection.

The first term is $4x(2x)$, or $8x^2$.
The second term is $1(2x) + 4x(3)$, or $+14x$.
The third term is $1(3)$, or $+3$.
The product is $8x^2 + 14x + 3$.

Compare this with the process written out in detail.

$$\begin{aligned}(4x + 1)(2x + 3) &= (4x + 1)(2x) + (4x + 1)(3)\\ &= 8x^2 + 2x + 12x + 3\\ &= 8x^2 + (2 + 12)x + 3\\ &= 8x^2 + 14x + 3\end{aligned}$$

Now try these

1. In Exercises 2–10, the first and last terms of the product are given. In each exercise, how is the first term found? How is the last term found? Which terms of the factors will you multiply and then add to find the middle term?

Replace ● by + or −, and find the middle term in each product.

2. $(5a - 3)(2a - 2) = 10a^2 \bullet \underline{?} + 6$
3. $(4a - 5)(5a - 1) = 20a^2 \bullet \underline{?} + 5$
4. $(8x^2 - 3)(4x^2 + 5) = 32x^4 \bullet \underline{?} - 15$
5. $(2ab - 3)(3ab + 3) = 6a^2b^2 \bullet \underline{?} - 9$
6. $(5a - 3b)(2a + 3b) = 10a^2 \bullet \underline{?} - 9b^2$
7. $(6a + 2b)(9a - b) = 54a^2 \bullet \underline{?} - 2b^2$
8. $(3x - 2y)(2x - 5y) = 6x^2 \bullet \underline{?} + 10y^2$
9. $(a - b)(a + b) = a^2 \bullet \underline{?} - b^2$
10. $(x - 5)(x + 5) = x^2 \bullet \underline{?} - 25$

Answers: **1.** Product of first terms of each factor; product of last terms of each factor; the two inner terms and the two outer terms. **2.** $-16a$ **3.** $-29a$ **4.** $+28x^2$ **5.** $-3ab$ **6.** $+9ab$ **7.** $+12ab$ **8.** $-19xy$ **9.** ± 0 **10.** ± 0

Exercises

Any method (FOIL)

A Write the products in polynomial form by inspection. Verify that you are correct in Exercises 1–6 by substituting -3 for x.

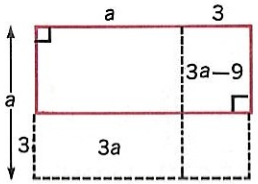
1. $(x + 5)(x + 3)$
2. $(x - 5)(x + 7)$
3. $(x + 2)(x - 2)$
4. $(x + 3)(x + 2)$
5. $(3x + 2)(5x + 7)$
6. $(3x + 4)(2x - 3)$
7. $(a - 2)(a + 3)$
8. $(y - 5)(y + 1)$

9. $(b - 7)(b + 3)$
10. $(2x + 5)(5x + 6)$
11. $(d + 4)(d - 5)$
12. $(m + n)(m - n)$
13. $(3y + 2)(4y + 3)$
14. $(2x + 5)(3x + 1)$
15. $(x + 3)(2x + 7)$
16. $(p + 9)(p + 11)$
17. $(3x - 4)(3x + 4)$
18. $(m - n)(p - q)$
19. $(b + 4)(b + 4)$
20. $(n - 2)(n + 2)$
21. $(m + 7)(m - 11)$
22. $(a + b)(2a - b)$
23. $(a - 2c)(a + 2c)$
24. $(3x - 2y)(3x + 2y)$
25. $(\frac{1}{2}x + 5)(6x - 10)$
26. $(\frac{1}{2}x - 3)(7x + 8)$
27. $(5a + 1)(\frac{1}{2}a + 4)$
28. $(\frac{1}{3}t + 8)(\frac{1}{4}t + 6)$
29. $(5x + \frac{2}{3})(6x + \frac{3}{2})$
30. $\left(\frac{x}{5} - 7\right)\left(\frac{x}{7} - 5\right)$
31. $(0.3u - 0.4)(0.5u - 0.1)$
32. $(1.8w - 2.4)(0.2u + 0.5)$
33. $(3x + 2)(x - 1)$
34. $(0.4m + 0.3n)(1.1m + 1.2n)$
35. $(x^2 + 2y)(x^2 - 2y)$
36. $(y^2 + 3)(y^2 + 3)$
37. $(2b + 3)(4b - 7)$
38. $(4y - 3)(4y - 3)$
39. $(4y - 3)(4y + 3)$
40. $(2a + 5)(2a + 3)$
41. $(4c - 7)(5c + 9)$
42. $(5x + 3)(x + 1)$
43. $(2f + 9)(3f + 7)$
44. $(7b + 2)(5b - 1)$
45. $(10p + 3)(6p - 1)$
46. $(8x + 3)(3x - 4)$
47. $(8r - 5)(7r + 5)$
48. $(2c - 9)(3c - 8)$
49. $(4x + 7)(5x - 3)$
50. $(x^2 - 4)(x^2 + 4)$
51. $(y^2 - 2)(y^2 + 2)$
52. $(7a - 2)(a + 8)$
53. $(a^2 - 3b)(a^2 + 2b)$
54. $(3x - 5y)(2x + 3y)$

B
55. $(\sqrt{2} + x)(\sqrt{2} - x)$
56. $(3^{\frac{1}{2}} - y)(3^{\frac{1}{2}} + y)$
57. $(\sqrt{2} + x)(\sqrt{2} + 2x)$
58. $(-3\sqrt{2} - y)(-4\sqrt{2} - y)$
59. $(\sqrt{2}x - 3)(\sqrt{2}x - 4)$
60. $(3^{\frac{1}{2}}a - b)(3^{\frac{1}{2}}a + 2b)$

C
61. $(3 - x^{\frac{1}{2}})(3 - x^{\frac{1}{2}})$
62. $(3\sqrt{2}x - 4)(-2\sqrt{2}x + 9)$
63. $(-\frac{7}{3}\sqrt{2} + x)(3\sqrt{2} - 2x)$
64. $(4\sqrt{2}x + 3\sqrt{3})(5\sqrt{2}x + 3\sqrt{3})$

65. Explain, using the areas of rectangular regions, how the drawing shows that $(a + 3)(a - 3) = a^2 - 9$ is true.



7–5 Squaring Binomials by Inspection

Multiply to see that each of the following is a true sentence.

a. $(a-5)^2 = (a-5)(a-5) = a^2 - 10a + 25$

b. $(2x+7)^2 = (2x+7)(2x+7) = 4x^2 + 28x + 49$

c. $(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$

d. $(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$

Look again at **c** and **d**. To square $x+y$, you square x, getting x^2, then take twice the product of x and y, getting $+2xy$, and then square y, getting $+y^2$. The square is the sum $x^2 + 2xy + y^2$. To square $a-b$, you square a, getting a^2, then take twice the product of a and $-b$, getting $-2ab$, and then square $-b$, getting $+b^2$. The square is $a^2 - 2ab + b^2$.

By following the pattern, you can square binomials by inspection.

① Square the first term of the binomial.

② Add to this twice the product of the first and last terms.

③ Add to this the square of the last term.

The shortcut for writing the square of a binomial can be used to square numbers also.

EXAMPLE 1. Square 32.

$$32^2 = (30+2)^2 = 30^2 + 2(30 \cdot 2) + 2^2$$
$$= 900 + 120 + 4 = 1024$$

EXAMPLE 2. Compute the square of 49.

$$49^2 = (50-1)^2 = 2500 - 100 + 1 = 2401$$

The advantage is that much of the work can be done "in your head."

Exercises

inspection

A — Write the square as a trinomial by inspection. Verify that your answer is correct in Exercises 1–6 by substituting $-\frac{1}{2}$ for a.

1. $(a+3)^2$	2. $(a-2)^2$	3. $(a-5)^2$
4. $(2a+1)^2$	5. $(4a-1)^2$	6. $(2a+3)^2$
7. $(5m-1)^2$	8. $(5m+3)^2$	9. $(7m-4)^2$
10. $(3x-y)^2$	11. $(x+2)^2$	12. $(n+3)^2$

13. $(x-y)^2$	14. $(x-3)^2$	15. $(a-5)^2$
16. $(n-6)^2$	17. $(2x-3)^2$	18. $(2x-5)^2$
19. $(2x+y)^2$	20. $(y+10)^2$	21. $(2x-3y)^2$
22. $(3a-2b)^2$	23. $(10a+2)^2$	24. $(5c+3d)^2$
25. $(2x^2+3y^2)^2$	26. $(5c^2+4y^3)^2$	27. $(a^2-3b^2)^2$
28. $(x^2y-5)^2$	29. $(a^2-3)^2$	30. $(2a^2+3)^2$

— Name with a decimal numeral. Use your pencil to write only the answers.

31. $(20+1)^2$	32. $(100-1)^2$	33. $(50+2)^2$
34. $(20-1)^2$	35. $(1000-1)^2$	36. $(100+5)^2$
37. 98^2	38. 101^2	39. 19^2

B — Each binomial below is a polynomial over the rational numbers. Write the square as a trinomial over the rational numbers by inspection.

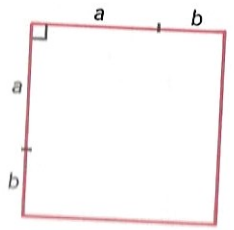
40. $(x+\frac{1}{2})^2$	41. $(x+\frac{1}{3})^2$	42. $(x-\frac{1}{6})^2$
43. $(2y+\frac{1}{2})^2$	44. $(6b+\frac{1}{2})^2$	45. $(5c+\frac{1}{3})^2$
46. $(9a+\frac{1}{6})^2$	47. $\left(x-\frac{a}{2}\right)^2$	48. $(y-\frac{1}{2}p)^2$
49. $\left(x+\frac{y}{2}\right)^2$	50. $\left(\frac{x}{2}+4\right)^2$	51. $(5b-\frac{1}{4})^2$

— Write as a trinomial over the real numbers.

52. $(x+\sqrt{3})^2$	53. $(2^{\frac{1}{2}}+m)^2$	54. $(3^{-\frac{1}{2}}-y)^2$
55. $(\sqrt{2}x+2)^2$	56. $(3x-\sqrt{11})^2$	57. $(\sqrt{2}x-\sqrt{2})^2$

— Find the area of the region enclosed in red.

58.



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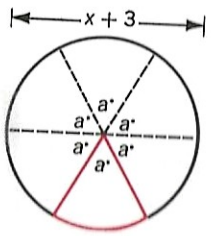
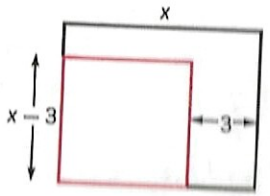


60.

61.



62. Examine the equations at the right and form a rule for squaring a number whose numeral ends in 5. Test the rule on 55, 25, and 15.

$85^2 = 7225$
$35^2 = 1225$
$45^2 = 2025$
$65^2 = 4225$

Now, treating the numbers as polynomials, find $(80 + 5)^2$, $(30 + 5)^2$, $(40 + 5)^2$, and $(60 + 5)^2$. Do you see why the rule you formulated works?

If t is the number of tens and 5 is the number of units, what is the number? What is its square? How does this prove that the rule works?

7–6 Product of $a + b$ and $a - b$

The product of binomials of the form $a + b$ and $a - b$ is easy to write by inspection. Note the examples below.

a. $(x + 7)(x - 7) = x^2 + 7x - 7x - 49 = x^2 - 49$
b. $(2x - 5)(2x + 5) = 4x^2 - 10x + 10x - 25 = 4x^2 - 25$
c. $(a + b)(a - b) = a^2 + ab - ab - b^2 = a^2 - b^2$

Can you see the pattern for finding the product of the sum and difference of two numbers? State it in your own words.

Now try these

— Find the following products.

1. $(x + 1)(x - 1)$
2. $(a + 3)(a - 3)$
3. $(2r - 2)(2r + 2)$
4. $(\frac{1}{2}y + 8)(\frac{1}{2}y - 8)$

Answers: **1.** $x^2 - 1$ **2.** $a^2 - 9$ **3.** $4r^2 - 4$ **4.** $\frac{1}{4}y^2 - 64$

Exercises

A — Find the following products. Verify that Exercises 1–6 are correct by replacing p with -5.

1. $(p + 7)(p - 7)$
2. $(p - 5)(p + 5)$
3. $(p + 8)(p - 8)$
4. $(2p + 5)(2p - 5)$
5. $(2p + 3)(2p - 3)$
6. $(3p + 5)(3p - 5)$
7. $(b + c)(b - c)$
8. $(c - 2)(c + 2)$
9. $(R - r)(R + r)$
10. $(2x + y)(2x - y)$
11. $(5x + 4y)(5x - 4y)$
12. $(7x - 6y)(7x + 6y)$

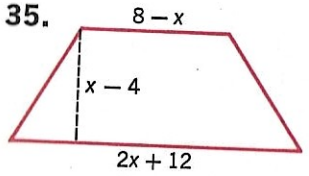
13. $(6x^2 - 7y)(6x^2 + 7y)$
14. $(a^2 + b^2)(a^2 - b^2)$
15. $(ab - c)(ab + c)$
16. $(y - x^2)(x^2 + y)$
17. $(\frac{1}{2} - 2x)(\frac{1}{2} + 2x)$
18. $(100 - 10)(100 + 10)$
19. $(2c + d)(2c - d)$
20. $(3c + 2d)(3c - 2d)$
21. $(5y - 4)(4 + 5y)$
22. $(x^2 + y^2)(x^2 + y^2)$
23. $(x^2 - y^2)(x^2 + y^2)$
24. $(x^2 - y^2)(x^2 - y^2)$

— Name with a decimal numeral. Use your pencils to write only the answer.

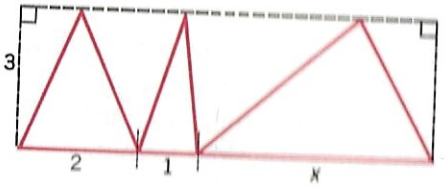
25. $(20 + 1)(20 - 1)$
26. $(30 + 1)(30 - 1)$
27. $(100 + 2)(100 - 2)$
28. $(50 + 4)(50 - 4)$
29. $(10 + 5)(10 - 5)$
30. $(1000 + 1)(1000 - 1)$
31. $18 \cdot 22$
32. $101 \cdot 99$
33. $51 \cdot 49$
34. $51 \cdot 29$

— Find the area of the region bounded by the red closed curve.

35.



36.



B — Name as a polynomial over the real numbers.

37. $(3\sqrt{2}x - \sqrt{5})(3\sqrt{2}x + \sqrt{5})$
38. $(2^{\frac{1}{2}}a - 3 \cdot 2^{\frac{1}{2}}b)(2^{\frac{1}{2}}a + 3 \cdot 2^{\frac{1}{2}}b)$
39. $(2^{\frac{1}{3}}m - 5)(2^{\frac{1}{3}}m + 5)$
40. $[3 \cdot (-6)^{\frac{1}{3}}m - 5][3 \cdot (-6)^{\frac{1}{3}}m + 5]$
41. $[(x + \sqrt{2}) + 5][(x + \sqrt{2}) - 5]$
42. $[5 - (2x + \sqrt{5})][5 + (2x + \sqrt{5})]$

43. Can you find a pattern in the equations at the right for multiplying a number ending in 5 by a number 10 greater?

$15 \times 25 = 375$
$35 \times 45 = 1575$
$85 \times 95 = 8075$
$55 \times 65 = 3575$

Verify that the answers are correct by naming the factors as the product of $a + b$ and $a - b$, for example, $15 \times 25 = (20 - 5)(20 + 5)$. Can you explain now why the rule works?

Let t be the total number of tens in the greater number and 5 the number of units. What is the number? What is the smaller number? Find their product to prove your rule.

7–7 Factors of Numbers

Since $35 = 7 \cdot 5$ is true, 7 is a **factor** of 35 and also 5 is a factor of 35. In general, if $c = a \cdot b$ is true, then a is a factor of c, and b is a factor of c.

Below are some of the ways that 18 can be named as the product of two factors.

Whole-number factors	Integral factors	Rational-number factors
$18 = 1(18)$	$18 = -1(-18)$	$18 = \frac{1}{2}(36)$
$18 = 2(9)$	$18 = 1(18)$	$18 = \frac{1}{10}(180)$
$18 = 3(6)$	$18 = -2(-9)$	$18 = 10(1.8)$
$18 = 6(3)$	$18 = 2(9)$	$18 = -100(-0.18)$
$18 = 9(2)$	$18 = -3(-6)$	$18 = \frac{18}{7}(\frac{7}{1})$
$18 = 18(1)$	$18 = 3(6)$	$18 = \frac{1}{6}(108)$

The set of factors from the first column is $\{1, 18, 2, 9, 3, 6\}$. Can you find any whole-number factor of 18 not in the set?

Can you find any integral factor of 18 not in the following set of factors from the second column?

$$\{-1, -18, 1, 18, -2, -9, 2, 9, -3, -6, 3, 6\}$$

Is the set of integral factors of 18 finite or is it infinite?

The rational-number factors of 18 that are in the third column form the following set.

$$\{\tfrac{1}{2}, 36, \tfrac{1}{10}, 180, 10, 1.8, -100, -0.18, \tfrac{18}{7}, \tfrac{7}{1}, \tfrac{1}{6}, 108\}$$

Can you think of rational factors of 18 that are not in the set? Is the set of rational factors of 18 finite or infinite?

When you are asked to find the factors of a number, you must know whether you are to find the whole-number, integral, rational, or real factors. For example, $\frac{1}{2}$ has no whole-number factors, but it has an infinite number of rational or real factors. The number 17 is prime because its only whole-number factors are 1 and 17, but it would not be correct to say that a prime number has exactly two factors. You must specify that there are exactly two whole-number factors.

When a whole number is named as the product of prime-number factors or powers of prime factors, the number is expressed in **prime-factorization form.** The examples at the top of the next page show how a whole number can be named in prime-factorization form.

Method I	Method II
$250 = 25 \cdot 10$	$250 = 2 \cdot 125$
$= 5 \cdot 5 \cdot 2 \cdot 5$	$= 2 \cdot 5 \cdot 5 \cdot 5$
$= 2 \cdot 5^3$	$= 2 \cdot 5^3$

Begin with any pair of whole-number factors greater than 1 and continue factoring until you have all primes. Then you have the prime-factorization form.

There is only one prime factorization for each composite whole number disregarding the order of the factors (recall that a **composite number** is any whole number greater than 1 that is not prime). This is the **unique factorization theorem.**

For negative numbers, the number is named as the product of -1 and the prime factors to get the prime-factorization form.

$$\begin{aligned} -125 &= -1(125) \\ &= -1(5 \cdot 5 \cdot 5) \\ &= -1(5^3) \end{aligned}$$

Exercises

A — What whole-number factor makes each sentence true?

1. $36 = 9x$ **2.** $18 = 3y$ **3.** $100 = 2m$

4. $75 = 5a$ **5.** $m \cdot m = 49$ **6.** $a \cdot 100 = 100$

— What rational-number factor makes each sentence true?

7. $\frac{1}{2}x = 10$ **8.** $(-3)x = 10$ **9.** $x \cdot x = 25$

10. $\frac{2}{3}x = \frac{5}{7}$ **11.** $\frac{1}{7}x = \frac{9}{15}$ **12.** $2\frac{1}{3}y = 6\frac{1}{9}$

— What integral factor, if any, makes each sentence true?

13. $-2m = 36$ **14.** $7y = -56$ **15.** $-3a = -30$

16. $12a = -60$ **17.** $5a = 26$ **18.** $-17m = 17$

19. $-56q = 280$ **20.** $-7m = 0$ **21.** $3t = -204$

— Name each number in prime-factorization form.

22. 36	**23.** 18	**24.** 21	**25.** 15	**26.** 42
27. 32	**28.** 45	**29.** 49	**30.** 9	**31.** 25
32. 28	**33.** 33	**34.** 44	**35.** 56	**36.** 396
37. 425	**38.** 500	**39.** 625	**40.** 1000	**41.** 960

B — What real-number factor makes each sentence true?

42. $y\sqrt{5} = 5$ **43.** $16 = m \cdot 2$ **44.** $30 = x\sqrt{5}$

45. $8^{\frac{1}{3}}r = 8$ **46.** $7^{\frac{1}{4}}y = -7$ **47.** $4^{\frac{1}{2}}t = 6$

48. $a\sqrt{3} = 9$ **49.** $v\sqrt[3]{5} = 5$ **50.** $k\sqrt[4]{2} = 2$

7–8 Monomial Factors of Polynomials

When you used the distributive postulate to find products, you wrote $a(b + c) = ab + ac$. You can also use the distributive postulate for factoring by using the symmetric property of equality, $ab + ac = a(b + c)$. The polynomial $ab + ac$ has the factors a and $b + c$. A factor such as a is a **common monomial factor** in the terms ab and ac. Look at these examples.

Finding products	Finding factors
$a(b + c) = ab + ac$	$ab + ac = a(b + c)$
$6(x + y) = 6x + 6y$	$6x + 6y = 6(x + y)$
$a(b + c + d) = ab + ac + ad$	$ab + ac + ad = a(b + c + d)$
$5x(3x^2-4x-1) = 15x^3-20x^2-5x$	$15x^3-20x^2-5x = 5x(3x^2-4x-1)$

In $6x + 6y$, 6 is the common monomial factor, and $x + y$ is the other factor. Notice that you usually do not factor the number when you are factoring a polynomial. You do not need to write $6x + 6y$ as $2 \cdot 3(x + y)$.

When you factor polynomials over the integers, it is customary to name the given polynomial as the product of two polynomials over the integers. You would not factor $6x + 6y$ as $12\left(\frac{x}{2} + \frac{y}{2}\right)$. If a polynomial does not have a polynomial factor over the integers, except 1, it is a **prime polynomial** with respect to the integers. For example, $6x + 5y$ is prime with respect to the integers and would not be factored as $5(\frac{6}{5}x + y)$.

Exercises

A — One monomial factor of each polynomial is written, but one or more terms of the other factor have been left out. Write the complete second factor.

1. $3x + 6y = 3(\_? + 2y)$
2. $4n - 2 = 2(2n - \_?)$
3. $3b^2 - 24b = 3b(\_? - 8)$
4. $15ab^2 - 3b^2 = 3b^2(\_? - \_?)$
5. $-3ax + 6ay = -3a(\_? - \_?)$
6. $p^2 - p = p(\_? - \_?)$
7. $8y - 24 = 8(y - \_?)$
8. $8x^2y^2 - xy^2 = xy^2(\_? - \_?)$
9. $x^2 - 3x^2y = x^2(\_? - \_?)$
10. $6a^2b - 3ab^2 = 3ab(\_? - \_?)$
11. $3x^2 - 6xy + 9y = 3(\_? - \_? + \_?)$
12. $-xy + 6x^2y - 4xy = -xy(\_? - \_? + \_?)$
13. $8x + 8y + 2xy = 2(\_? + \_? + \_?)$
14. $-5x^3 - 10y^3 = -5(\_? + \_?)$
15. $4xy - 2x - 3x^2y = x(\_? - \_? - \_?)$

One factor is given for each polynomial. Write the other factor. Check your factors by multiplying by inspection.

16. $5x + 20 = \_?(x + 4)$
17. $12y - 4 = \_?(3y - 1)$
18. $16m - 4m^2 = \_?(4 - m)$
19. $400x - x^2 = \_?(400 - x)$
20. $16x^3 - 2x = 2x(\_? - \_?)$
21. $4ab - b = b(\_? - \_?)$
22. $4ab + b = b(\_? + \_?)$
23. $25x^2 - 20y^2 = \_?(5x^2 - 4y^2)$
24. $2mn - n = n(\_? - \_?)$
25. $7x + 49xy = 7x(\_? + \_?)$
26. $1000x^3 - x = x(\_? - \_?)$
27. $a^3 + 2a^2b = a(\_? + \_? + \_?)$
28. $8x^2 - 28x - 4 = \_?(2x^2 - 7x - 1)$
29. $5m - 10mn - 15 = \_?(m - 2mn - 3)$
30. $4x^3 + 6x^2 - 2x = 2x(\_? + \_? - \_?)$

Factor the following polynomials over the integers. One factor should be the most inclusive factor common to all terms.

Example 1. $3x + 3y$

The most inclusive factor common to both terms is 3.

$$3x + 3y = 3(x + y)$$

Example 2. $7x^3 - 21x$

The most inclusive factor common to both terms is $7x$.

$$7x^3 - 21x = 7x(x^2 - 3)$$

Note that $7x^3 - 21x$, $7x$, and $x^2 - 3$ are all polynomials over the integers.

31. $7x + 7y$
32. $3a + 9b$
33. $8x - 4y$
34. $ax + bx$
35. $3x - am$
36. $9x^2 - 6y^2$
37. $4c + ac + bc$
38. $ay + by + 3y$
39. $d^2 + 2d + ad$
40. $ax - ay + a^2$
41. $5x - 10y + 15z^2$
42. $5m + 10m^2 + m^3$
43. $2\pi r^2 + 2\pi rh$
44. $a^2b + ab^2 + a^3b^3$
45. $4a^2 + 12ab + 4b^2$

B — Some of the following polynomials have monomial factors over the integers. Factor those that have common monomial factors, and write *Prime polynomial* for those that do not.

46. $3a + b$
47. $7a^2b - 7ab$
48. $12n^2 - 4 + 16n$
49. $8x^2 - 3y^2$
50. $36x^2 - 18xy$
51. $14x^2 + 7xy + 21y^2$

52. $7n^3 - 8nm + 3n^2$
53. $9ab + a^2 + 2b^2$
54. $39a^3 - 52ab^2 + 65ac^2$
55. $27x^4 + 3x^3 + 9ax$
56. $9s - 13s^2 + 5t$
57. $25xy - 50xz + 100x$
58. $4x^2 - 8x + 6$
59. $3x^2 + 6xy - 2x^3$
60. $a^3 + 3a^2b + 3ab^2$
61. $14 - 42p + 7$
62. $3ab^2 - ab + 2a^2$
63. $8x^2 - 12xy - 16xz$
64. $ay^2 + ab + 3a$
65. $3x^3 - 15x^2 - 6x$
66. $8a^3 - 4ax^2 - 12a^2x^2$
67. $3a^2 + 4a^3 - 5a^4b$
68. $18ab^3c - 36a^2bc^3$
69. $4x^2y - 12xyz + 8xz^2$

7–9 Factoring the Difference of Two Squares

In a previous section, you found a pattern that led to the following products.

$$(a + b)(a - b) = a^2 - b^2$$
$$(4x + y)(4x - y) = 16x^2 - y^2$$
$$(5x - 3y)(5x + 3y) = 25x^2 - 9y^2$$

Explain why the polynomial form in each of these illustrations is called the difference of two squares.

You can reverse this pattern by the symmetric property of equality to find a pattern for factoring the difference of two squares.

Since the first term of the difference of two squares is the square of some expression, the first term of each factor must be the square root of the first square. Similarly, the second term of each factor is the square root of the second square. One factor must have + between the terms, and the other must have −, eliminating the possibility of a middle term in the polynomial.

EXAMPLE. Factor $25x^2 - 36y^2$.

① This is a difference of two squares, so you find the square root of the first term. $(5x \quad)(5x \quad)$

② Next, you find the square root of the second term. $(5x \quad 6y)(5x \quad 6y)$

③ Now fill in the signs. One has a plus sign, the other has a minus sign. Although it does not matter which comes first, you must have one of each. $(5x + 6y)(5x - 6y)$ or $(5x - 6y)(5x + 6y)$

Can the *sum* of two squares, say $a^2 + b^2$, be factored in the same manner? Rewrite $a^2 + b^2$ as $a^2 - (-b^2)$. Following the pattern for the difference of two squares, the factors would be $(a + \sqrt{-b^2})$ and $(a - \sqrt{-b^2})$. But b^2 is a positive number, so $-b^2$ is negative. Since a negative number does not have a real square root, $-b^2$ is not a real number. The *sum of two squares cannot be factored* over the set of real numbers.

Now try these

— Factor each of the following polynomials.

1. $x^2 - 16$ **2.** $4r^2 - 1$ **3.** $9 - y^2$ **4.** $\frac{m^2}{4} - \frac{1}{4}$

Answers: **1.** $(x - 4)(x + 4)$ **2.** $(2r - 1)(2r + 1)$ **3.** $(3 - y)(3 + y)$ **4.** $\left(\frac{m}{2} - \frac{1}{2}\right)\left(\frac{m}{2} + \frac{1}{2}\right)$

Exercises

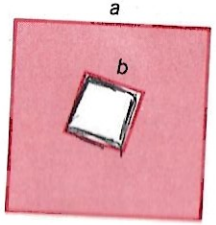
A — Factor each of the following polynomials over the integers. Write *Prime polynomial* for each that cannot be factored over the integers.

1. $y^2 - 9$ **2.** $x^2 - 4$ **3.** $x^2 - 25$
4. $y^2 - 36$ **5.** $4x^2 - 9$ **6.** $9x^2 - 4$
7. $16y^2 - 36$ **8.** $-64 + 4x^2$ **9.** $x^2y^2 - a^2$
10. $y^6 - 36$ **11.** $9x^4y^2 - b^2$ **12.** $1 - 81y^2$
13. $x^2y^2 - z^2$ **14.** $-25 + 4x^2y^2$ **15.** $x^2 + y^2$
16. $x^2 - y^4$ **17.** $4x^2 + 16y^2$ **18.** $81x^2 - 9y^2$

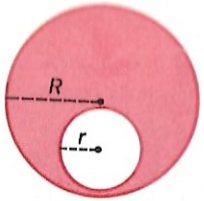
— Factor each polynomial over the rational numbers.

19. $\frac{1}{9}x^2 - 16$ **20.** $\frac{25}{36}a^2 - b^2$ **21.** $\frac{4}{9}x^2 - \frac{25}{36}y^2$
22. $\frac{121}{9}r^2 - s^2$ **23.** $-9 + x^2$ **24.** $-\frac{1}{16} + r^2$
25. $-25 + \frac{1}{64}b^2$ **26.** $\frac{a^2}{9} - \frac{b^4}{16}$ **27.** $\frac{4t^2}{49} - \frac{v^4}{81}$

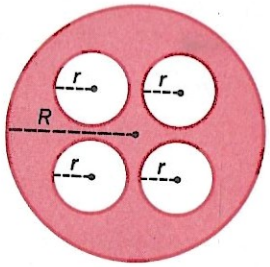
28. The figure at the right shows a large square whose side is a units. Within the large square is a small square (in any position) whose side is b units. What is the area of the large square? What is the area of the small square? What is the area of the shaded surface between the two squares? Factor the expression for the shaded area.



29. A small circle whose radius is r units is drawn within a large circle whose radius is R units. What is the area of the large circle? What is the area of the shaded part? Factor the expression for the shaded area.



30. If four circular holes (radius r) are cut in a large circular plate of radius R, what is the area of the remaining surface in terms of R and r? Factor the expression for the shaded area.



B — Factor each polynomial over the real numbers.

Example. $x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$

31. $y^2 - 7$ **32.** $x^2 - 11$ **33.** $a^2 - 27$

34. $x^2 - 50$ **35.** $36 - m^2$ **36.** $27 - r^2$

37. $150 - q^2$ **38.** $1 - 2x^2$ **39.** $t^2 - 5v^2$

— Factor each polynomial over the integers. (*Hint:* In Exercise 40, think of $a + b$ as one number x.)

40. $(a + b)^2 - c^2$ **41.** $c^2 - (a + b)^2$

42. $(r + s)^2 - t^2$ **43.** $(r + s)^2 - (t + u)^2$

44. $(x - a)^2 - (y - b)^2$ **45.** $4(x - a)^2 - 9(y + b)^2$

7–10 Factoring a Perfect Square Trinomial

A **perfect square trinomial** is the square of a binomial. Thus, $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ are perfect square trinomials because they are the squares of $a + b$ and $a - b$, respectively.

The characteristics of a perfect square trinomial may be summarized as follows.

① The first and third terms are perfect squares.

② The absolute value of the middle term is twice the product of the principal square roots of the first and third terms.

Having recognized a perfect square trinomial, you find its two **binomial factors** by reversing the process of squaring the binomial. The square root of the first term of the trinomial is the first term of the binomial factors. A square root of the third term of the trinomial is the second term of the binomial factors. The sign between the terms is determined by the signs of the trinomial. When the signs of the trinomial are all +, the sign in the binomial factor is +. When the signs of the terms alternate in the order +, −, +, the sign in the binomial is −.

EXAMPLE. Factor $4x^2 + 12x + 9$.

Note that the given polynomial follows the pattern described above for a perfect square trinomial. The terms $4x^2$ and 9 are perfect squares; $|12x|$ is equal to $2 \cdot \sqrt{4x^2} \cdot \sqrt{9}$. The first term of the trinomial is $4x^2$, so the square root of $4x^2$, or $2x$, is the first term of the binomial factor. The square root of the last term of the trinomial, 9, is 3. The signs are +, +, +, so the sign in the binomial factor is also +.

$(2x \quad)(2x \quad)$
$(2x \quad 3)(2x \quad 3)$
$(2x + 3)(2x + 3)$
or
$(2x + 3)^2$

Examine the following table for other examples.

Perfect square trinomial	Factors		Factored expression
$x^2 + 2xy + y^2$	$x + y$	$x + y$	$(x + y)^2$
$x^2 - 2xy + y^2$	$x - y$	$x - y$	$(x - y)^2$
$4x^2 - 12x + 9$	$2x - 3$	$2x - 3$	$(2x - 3)^2$
$x^4 + 2x^2y^2 + y^4$	$x^2 + y^2$	$x^2 + y^2$	$(x^2 + y^2)^2$

Exercises

A — Each of the following trinomials is a perfect square. Write each one as the square of a binomial. One term of the binomial is written for you.

1. $a^2 + 4ab + 4b^2 = (a + \underline{?})^2$

2. $x^2 + 2xy + y^2 = (\underline{?} + y)^2$

3. $4x^2 - 4x + 1 = (2x - \underline{?})^2$

4. $b^2 - 8bc + 16c^2 = (b - \underline{?})^2$

5. $s^2 + 6st + 9t^2 = (s + \underline{?})^2$

6. $n^2 - 12n + 36 = (\underline{?} - 6)^2$

7. $4x^2 + 20x + 25 = (2x + \underline{?})^2$

8. $64x^2 - 16xy + y^2 = (\underline{?} - y)^2$

9. $25 - 10y + y^2 = (\underline{?} - y)^2$ **10.** $9a^2 - 24a + 16 = (\underline{?} - 4)^2$

11. $1 + 8b + 16b^2 = (\underline{?} + 4b)^2$ **12.** $a^2 - 6ab + 9b^2 = (\underline{?} - 3b)^2$

— Replace ● with + or − to make the square of the binomial equal the trinomial.

13. $n^2 + 14n + 49 = (n \bullet 7)^2$ **14.** $x^2 - 16x + 64 = (x \bullet 8)^2$

15. $x^2 - 10x + 25 = (x \bullet 5)^2$ **16.** $p^2 - 10pq + 25q^2 = (p \bullet 5q)^2$

17. $a^2 + 8a + 16 = (a \bullet 4)^2$ **18.** $n^2 - 12n + 36 = (n \bullet 6)^2$

19. $a^2 - 16a + 64 = (a \bullet 8)^2$ **20.** $4x^2 - 12x + 9 = (2x \bullet 3)^2$

21. $b^2 - 6b + 9 = (b \bullet 3)^2$ **22.** $9 + 12x + 4x^2 = (3 \bullet 2x)^2$

— Write each of these perfect square trinomials as the square of a binomial. Check by squaring the binomial mentally.

23. $x^2 + 6x + 9$ **24.** $x^2 - 6x + 9$

25. $4x^2 - 12xy + 9y^2$ **26.** $4x^2 + 12xy + 9y^2$

27. $1 + 8x + 16x^2$ **28.** $1 - 8x + 16x^2$

29. $4x^2 - 4x + 1$ **30.** $4x^2 + 4x + 1$

31. $9y^2 + 12y + 4$ **32.** $9y^2 - 12y + 4$

33. $4a^2 - 4a + 1$ **34.** $4a^2 + 4a + 1$

35. $49a^2 - 84a + 36$ **36.** $4m^2 - 12m + 9$

37. $x^2 - 18x + 81$ **38.** $a^2 + 14a + 49$

39. $1 - 6b + 9b^2$ **40.** $4 - 20b + 25b^2$

B — Factor over the integers.

Example. $(x^2 + 2xy + y^2) - z^2 = (x + y)^2 - z^2$
$= (x + y - z)(x + y + z)$

41. $(a^2 + 2ab + b^2) - c^2$ **42.** $c^2 - (a^2 + 2ab + b^2)$

43. $(x^2 - 2xy + y^2) - 9$ **44.** $16 - (x^2 - 2xy + y^2)$

45. $a^2 - 9 + 2ab + b^2$ (*Hint:* First rearrange the terms.)

46. $x^2 - 16 - 2xy + y^2$ **47.** $(x + y)^2 - (a^2 + 2ab + b^2)$

C — Factor over the real numbers.

Example. $m^2 - 2\sqrt{11}m + 11 = (m - \sqrt{11})(m - \sqrt{11})$, or $(m - \sqrt{11})^2$

48. $a^2 + 2\sqrt{2}a + 2$ **49.** $x^2 - (y^2 + 2\sqrt{2}y + 2)$

50. $x^2 - 2\sqrt{3}x + 3$ **51.** $(x^2 - 2\sqrt{5}x + 5) - (y^2 + 2\sqrt{7}y + 7)$

52. In factoring the difference of two squares, what happens if you choose the negative square root of the first term? Write $(-a + b)(-a - b)$ in polynomial form.

53. Note that $(-a + b) = -(a - b)$ is true and $(-a - b) = -(a + b)$ is true. Can you explain the result of Exercise 52 using this idea?

54. In factoring a perfect square trinomial, what happens if you choose the negative square root of the first term? Is $(-a + b)^2$ equivalent to $(a + b)^2$? Is $(-a - b)^2$ equivalent to $(a + b)^2$?

— Factor each of the following in two ways.

55. $x^2 + 16x + 64$ **56.** $4t^2 - 49$ **57.** $36v^2 - 12v + 1$

7–11 Factoring $x^2 + (a + b)x + ab$

Consider the product of $x + a$ and $x + b$, where a and b are any real numbers.

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

Reversing the equation above by the symmetric property of equality, you can see that trinomials of the form $x^2 + (a + b)x + ab$ can be factored by finding two factors of the constant term whose sum is the numerical coefficient of the middle term, and adding them to x.

EXAMPLE 1. Factor $x^2 + 5x + 6$.

The set of factors of 6 is $\{1, 2, 3, 6\}$. Thus, the constant term could be $6 \cdot 1$ or $2 \cdot 3$.

Of these, the sum of one pair must give the coefficient of the middle term, 5. The sum of 2 and 3 is 5. Therefore, 2 and 3 are the second terms of the binomials.

Since the trinomial is of the form +, +, +, the sign in each binomial factor is +.

$(x \quad)(x \quad)$

$(x \quad 2)(x \quad 3)$

$(x + 2)(x + 3)$

EXAMPLE 2. Factor $x^2 - 5x - 24$.

The integral factors of −24 form the set

$\{1, -1, 2, -2, 3, -3, 4, -4, 6, -6, 8, -8, 12, -12, 24, -24\}$.

You do not need to write out the set. Of the factors of −24, the only pair whose product is −24 and whose sum is −5 is −8 and 3. These are the second terms of the binomial factors.

$(x \quad)(x \quad)$

$(x + (-8))(x + 3)$

$(x - 8)(x + 3)$

Explain why the signs in the binomials will be + and − whenever the sign of the last term of the trinomial is −.

In the following table you may examine further examples of factoring expressions of the form $x^2 + (a + b)x + ab$.

Trinomial	a, b	Factors	Factored expression
$x^2 + 3x + 2$	2, 1	$x + 1$ $x + 2$	$(x + 1)(x + 2)$
$x^2 + x - 30$	6, −5	$x + 6$ $x - 5$	$(x + 6)(x - 5)$
$x^2 - x - 30$	−6, 5	$x - 6$ $x + 5$	$(x - 6)(x + 5)$
$x^2 - 11x + 30$	−6, −5	$x - 6$ $x - 5$	$(x - 6)(x - 5)$

If the signs of a trinomial are +, −, +, in that order, what can you say about the signs in the binomial factors?

Some trinomials may be factored over the integers and some may not. For example, $x^2 - 2x - 24$ may be factored over the integers, but $x^2 + 3x + 1$ is prime with respect to the integers.

Now try these

— For each of the following, what two integers will have the first number as their sum and the second number as their product?

1. 7, 12 **2.** −5, 6 **3.** 5, −24 **4.** −6, 5
5. −6, 9 **6.** −2, −15 **7.** −9, 18 **8.** 4, −32
9. −4, −32 **10.** −1, −72 **11.** 1, −72 **12.** −13, 36

Answers: **1.** 3, 4 **2.** −2, −3 **3.** −3, 8 **4.** −5, −1 **5.** −3, −3 **6.** −5, 3 **7.** −3, −6 **8.** −4, 8 **9.** 4, −8 **10.** 8, −9 **11.** −8, 9 **12.** −4, −9

Exercises

A — Factor each of the following expressions.

1. $x^2 + 7x + 12$ $(x+4)(x+3)$ **2.** $x^2 - 5x + 6$ $(x-3)(x-2)$
3. $x^2 + 5x - 24$ $(x+8)(x-3)$ **4.** $x^2 - 6x + 5$ $(x-5)(x-1)$
5. $x^2 + 6x + 5$ $(x+5)(x+1)$ **6.** $x^2 - 2x - 15$ $(x-5)(x+3)$
7. $x^2 - 4x - 32$ $(x+4)(x-8)$ **8.** $x^2 - 13x + 36$ $(x-9)(x-4)$
9. $x^2 + 4x - 32$ $(x+8)(x-4)$ **10.** $x^2 - 9x + 18$ $(x-6)(x-3)$
11. $x^2 + x - 72$ $(x+9)(x-8)$ **12.** $x^2 - x - 72$ $(x-9)(x+8)$
13. $x^2 + 10x + 21$ $(x+7)(x+3)$ **14.** $x^2 - 6x + 8$ $(x-4)(x-2)$
15. $x^2 + 14x + 48$ $(x+8)(x+6)$ **16.** $x^2 - 8x + 15$ $(x-5)(x-3)$
17. $x^2 + 6x + 8$ $(x+4)(x+2)$ **18.** $x^2 + 10x + 16$ $(x+8)(x+2)$
19. $x^2 + 2x - 8$ $(x+4)(x-2)$ **20.** $x^2 - 5x - 14$ $(x-7)(x+2)$
21. $y^2 + 9y + 18$ $(y+6)(y+3)$ **22.** $y^2 - 17y + 66$ $(y-11)(y-6)$
23. $x^2 + 2x - 15$ $(x+5)(x-3)$ **24.** $y^2 - y - 156$ $(y-13)(y+12)$

25. $x^2 + x - 56$ $(x+8)(x-7)$ **26.** $y^2 + 6y - 72$ $(y+12)(y-6)$
27. $3 - 4t + t^2$ $(3-t)(1-t)$ **28.** $20 + 12v + v^2$ $(10+v)(2+v)$
29. $-30 + 13w + w^2$ $(10+w)(3+w)$ **30.** $51 - 20k + k^2$ $(17-k)(3-k)$

B — The following are the areas of rectangular regions expressed in terms of x, where x is an integer. If the length and width of each rectangle are also integers, find the dimensions in terms of x.

31. $x^2 - 169$ $(x-13)(x+13)$ **32.** $x^2 + 13x + 12$ $(x+12)(x+1)$
33. $x^2 - 13x + 12$ $(x-12)(x-1)$ **34.** $x^2 - 4x + 4$ $(x-2)(x-2)$

— For each of the following, first factor out a common monomial, then factor the remaining trinomial into its binomial factors.

Example. $2x^2 - 10x + 12 = 2(x^2 - 5x + 6) = 2(x - 2)(x - 3)$

35. $3x^2 + 6x - 24$ **36.** $4x^2 - 32x + 60$
37. $2x^2 + 12x - 80$ **38.** $5y^2 - 45y - 110$
39. $2t^2 - 28t + 98$ **40.** $3x^3 - 33x^2 + 84x$

C **41.** For each of Exercises 31–34, determine the dimensions of the smallest rectangle that meets the other conditions.

7–12 Factoring $ax^2 + bx + c$

To factor a trinomial completely is to obtain any common monomial factors and the binomial factors (if the trinomial is not prime). For example, $2x^2 + 10x + 12$ may be factored as $(2x + 6)(x + 2)$ or as $(x + 3)(2x + 4)$, but it is only factored completely as $2(x + 3)(x + 2)$.

A trinomial such as $2x^2 + 7x + 5$ has no common monomial factor, and it is not in any of the forms discussed earlier. Yet it is not prime over the integers. In this case the first terms of the binomial factors must be $2x$ and x to produce a product of $2x^2$. The last terms must be 5 and 1 to produce a product of 5. Since the form is +, +, +, the sign in each binomial will be +. But it is not finished yet, because there are two different possible arrangements and only one of them is correct. The product of the correct factors should give a middle term of $7x$.

$(2x \quad)(x \quad)$

Only one of
$(2x + 1)(x + 5)$
or
$(2x + 5)(x + 1)$

$$1 \cdot x + 2x \cdot 5 = 11x$$
$$5 \cdot x + 2x \cdot 1 = 7x$$

Hence, $2x^2 + 7x + 5$ is factored completely as $(2x + 5)(x + 1)$.

In the case just given, the choices were limited, but often the problem involves several more choices. Skill in factoring expressions of the form $ax^2 + bx + c$ comes through thinking about each problem logically and through obtaining a lot of practice. The following examples will help.

EXAMPLE 1. Factor $8x^2 - 35x + 12$.

The first term can be factored into $8x$ and x or into $4x$ and $2x$. Notice that -35, the numerical coefficient of the middle term, is odd. This means that you will need some odd coefficients in the binomials to give two numbers whose sum will be an odd number. So start with $8x$ and x.

Since the pattern is +, −, +, both binomials will have −.

$(8x \quad)(x \quad)$
$(8x - \quad)(x - \quad)$

Possible pairs of factors for 12 are 1 and 12, 2 and 6, and 3 and 4. You can rule out 2 and 6 because they are both even. In fact, of the four combinations shown at the right, you can rule out the two in which odd numbers are multiplied by even numbers because you need an odd number in the middle term.

Only one of
$(8x - 1)(x - 12)$
$(8x - 12)(x - 1)$
$(8x - 3)(x - 4)$
$(8x - 4)(x - 3)$

Of the last possibilities, it is clear that $8x \cdot 12$ will give too large a middle term, so the logical answer is $(8x - 3)(x - 4)$. Check by multiplying.

Only one of
$(8x - 1)(x - 12)$
$(8x - 3)(x - 4)$

$(8x - 3)(x - 4)$

EXAMPLE 2. Factor $8x^2 + 10x - 3$.

The case is somewhat different. The pattern is +, +, −, so the signs in the binomials will be + and −. It is usually best to put the signs in last when that happens. Since 3 has only odd factors and the middle term is even, $8x^2$ must be factored into $4x$ and $2x$. The factors of 3 are just 1 and 3. When these are tried, for $(4x \quad 1)(2x \quad 3)$, the products of the inner and outer terms will be $+2x$ and $-12x$ or $-2x$ and $+12x$. The other possibility gives $+6x$ and $-4x$ or $-6x$ and $+4x$. For the sum to be $+10x$, the choice must be $+12x$ and $-2x$ so the factored form of $8x^2 + 10x - 3$ is $(4x - 1)(2x + 3)$. Check by multiplying.

$(4x \quad)(2x \quad)$

Only one of
$(4x \quad 1)(2x \quad 3)$
or
$(4x \quad 3)(2x \quad 1)$

$(4x - 1)(2x + 3)$

When you are eliminating possibilities, you can do the work described here "in your head." With practice, you can soon learn which binomial factors are most likely for a given trinomial.

Now try these

Tell whether the result in each case is *Even* or *Odd*.

1. even × even
2. even × odd
3. even + even
4. odd × odd
5. even − even
6. odd + even
7. odd − even
8. odd + odd
9. odd − odd

Answers: 1. Even. 2. Even. 3. Even. 4. Odd. 5. Even. 6. Odd. 7. Odd. 8. Even. 9. Even.

Exercises

A Fill in the missing term or terms in the complete factorizations of the following trinomials.

1. $5x^2 + 9x - 18 = (5x - 6)(x + \underline{?})$
2. $4a^2 + 26a - 14 = 2(2a - \underline{?})(a + 7)$
3. $18n^2 + 21n - 4 = (6n - \underline{?})(3n + 4)$
4. $14x^2 - 13x - 12 = (7x + 4)(\underline{?} - 3)$
5. $40n^2 + n - 6 = (\underline{?} - 3)(\underline{?} + 2)$
6. $2a^2 + 7ab - 15b^2 = (2a - \underline{?})(a + \underline{?})$
7. $5a^2 + 2a - 7 = (5a + \underline{?})(a - \underline{?})$
8. $-3x^2 - 3x + 216 = -3(x + \underline{?})(x - \underline{?})$
9. $5(x^2 - 3x - 18) = 5(x - 6)(x + \underline{?})$
10. $3(2x^2 - 7x - 15) = 3(2x + \underline{?})(x - \underline{?})$

Factor completely each of the following trinomials.

11. $2x^2 + 11x + 4$
12. $3x^2 + 11x + 10$
13. $18y^2 + 6y - 4$
14. $15y^2 - y - 2$
15. $6y^2 - 17y + 12$
16. $42x^2 + 10x - 12$
17. $6x^2 + 5x - 4$
18. $9x^2 + 6x - 8$
19. $5x^2 - 16x + 3$
20. $8y^2 - 2y - 1$
21. $3y^2 + 4y - 7$
22. $x^2 - 4x - 32$

B Factor completely each of the following trinomials.

23. $8x^2 - 10xy + 3y^2$
24. $14x^2 - 57xy - 27y^2$
25. $18 - 9y - 35y^2$
26. $-21x^2 - 3x - 8$
27. $6x^2 + 21x + 18$
28. $40c^2 + 39cd - 40d^2$
29. $12x^2 - 29xy + 14y^2$
30. $56x^2 + 15x - 56$
31. $16a^2 + 56ab + 49b^2$
32. $3x^2 - 8xy - 3y^2$
33. $15x^4 - 10x^3 - 25x^2$
34. $18x^2 - 57x + 35$
35. $30s^2 + 39s - 9$
36. $4l^2 - 16l + 16$

7–13 Identifying Common Binomial Factors

Suppose you had to factor $a(b + c) + b(b + c)$. The binomial $(b + c)$ is a common factor of two terms. Think of $(b + c)$ as x.

$$a(b + c) + b(b + c) = ax + bx$$

Apply the distributive property.

$$ax + bx = (a + b)x$$

Therefore, $a(b + c) + b(b + c) = (a + b)(b + c)$ is true.

EXAMPLE 1. Factor $ax - xy - ac + cy$.

$$ax - xy - ac + cy = xa - xy - ca + cy$$

Apply distributive property. $= x(a - y) - c(a - y)$

Can you see why $-c$ is factored out of the last two expressions? Now let $(a - y)$ be k.

$$x(a - y) - c(a - y) = xk - ck, \text{ or } (x - c)k$$

Then go back to the original expression.

$$ax - xy - ac + cy = (x - c)(a - y)$$

EXAMPLE 2. Factor $4x - xy - 12 + 3y$.

$$\begin{aligned} 4x - xy - 12 + 3y &= 4x - yx - 12 + 3y \\ &= x(4 - y) - 3(4 - y) \\ &= (x - 3)(4 - y) \end{aligned}$$

Notice that it makes no difference if you group in another way.

$$\begin{aligned} 4x - 12 - xy + 3y &= 4(x - 3) - y(x - 3) \\ &= (4 - y)(x - 3) \end{aligned}$$

Exercises

A — Factor each of the following expressions.

1. $d(e + f) + g(e + f)$.
2. $x(b - c) - d(b - c)$
3. $a(b + 4) - c(b + 4)$
4. $x^2(a + b) + y^2(b + a)$
5. $rv + rw - sv - sw$
6. $km - kn - lm + ln$
7. $a(x + y) + b(x + y) + c(x + y)$
8. $ab + 5a - bc - 5c$
9. $10 + 2t - 5s - st$
10. $x^2(a + 3) - y^2(a + 3)$
11. $54g - 12gh - 45 + 10h$
12. $2xw - 6x + w - 3$

B — Factor each of the following expressions.

13. $\frac{2}{3}bc - \frac{14}{3}b + c - 7$
14. $\frac{3}{8}xy + \frac{3}{2}x - 8 - 2y$
15. $\sqrt{6} - 2s\sqrt{3} + r\sqrt{2} - 2sr$
16. $\frac{ac}{10} + \frac{3a}{5} - c - 6$
17. $2\sqrt{10} - 12ab - 3a\sqrt{2} + 8b\sqrt{5}$
18. $\frac{21}{5}x - \frac{3}{20}xy - 42 + \frac{3}{2}y$

Self-Test on Special Products and Factoring

The aim of this test is to see if you can correctly factor and multiply many different types of polynomials over the integers.

Cover *Column B*, factor the expression in *Column A*, and then check the answer in *Column B*.

Then cover *Column A*, multiply as indicated in *Column B*, and then check the answer in *Column A*.

Column A	Column B
1. $2a^2 + 14a + 24$	1. $2(a + 3)(a + 4)$
2. $5y^2 - 45$	2. $5(y + 3)(y - 3)$
3. $st^2 - st - 20s$	3. $s(t - 5)(t + 4)$
4. $7a^2 - 14a - 105$	4. $7(a - 5)(a + 3)$
5. $3x^2 + 12x + 45$	5. $3(x^2 + 4x + 15)$
6. $x^2 - 6x + 9$	6. $(x - 3)^2$
7. $6t^2 - 15t^3$	7. $3t^2(2 - 5t)$
8. $2 - 128t^2$	8. $2(1 + 8t)(1 - 8t)$
9. $ab^2 - ab - 72a$	9. $a(b - 9)(b + 8)$
10. $n^2 + 5n + 7$	10. Prime.
11. $3a^2 + a - 2$	11. $(3a - 2)(a + 1)$
12. $2a^2 - 5a + 3$	12. $(2a - 3)(a - 1)$
13. $a^2 + 25$	13. Prime.
14. $q^2 - 12q - 28$	14. $(q - 14)(q + 2)$
15. $2x^2 - 14x + 24$	15. $2(x - 3)(x - 4)$
16. $y^4 - 6y^2 - 16$	16. $(y^2 - 8)(y^2 + 2)$
17. $49c^2 + 70c + 25$	17. $(7c + 5)^2$
18. $5a^2 - 80$	18. $5(a + 4)(a - 4)$
19. $x^3 - x$	19. $x(x + 1)(x - 1)$
20. $a^4 - 16$	20. $(a + 2)(a - 2)(a^2 + 4)$
21. $1 - 4y^2$	21. $(1 + 2y)(1 - 2y)$
22. $x^2 + 4y^2$	22. Prime.
23. $3a^2 - a - 2$	23. $(3a + 2)(a - 1)$

7–14 Using Special Products to Solve Equations

Some equations are easily solved by using the special products you have learned because much of the work can be done mentally. Study the following examples.

EXAMPLE 1. Solve $(x-3)^2 = x^2 - 7x + 12$.

① Square $(x-3)^2$. $x^2 - 6x + 9 = x^2 - 7x + 12$

② Add $-x^2 + 7x$. $x + 9 = 12$

③ Add -9. $x = 3$

Check $(3-3)^2 = 3^2 - 7(3) + 12$

$0 = 9 - 21 + 12$

$0 = 0$

Solution set: $\{3\}$

EXAMPLE 2. Solve $2(x-5) - (x-2)(x+3) = 5 - (x-4)^2$.

① Multiply. $2x - 10 - (x^2 + x - 6) = 5 - (x^2 - 8x + 16)$

② Apply distributive property. $2x - 10 - x^2 - x + 6 = 5 - x^2 + 8x - 16$

③ Combine like terms. $-x^2 + x - 4 = -x^2 + 8x - 11$

④ Add $x^2 - 8x + 4$. $-7x = -7$

$x = 1$

Solution set: $\{1\}$

Notice that before the distributive property is applied, the product of the binomials is kept in parentheses to help avoid errors in signs.

Check the solution of Example 2.

Exercises

A — Solve for x.

1. $(x-3)^2 = x^2 - 5x - 8$
2. $(2x-3)^2 = 4x^2 - 6x - 3$
3. $(x+5)^2 = x^2$
4. $(x+3)^2 - (x+5)^2 = 10$
5. $4 + (x-9)^2 = 7 + x^2$
6. $(5x-6)^2 = (4x+1)^2 + 9x^2$
7. $(2x-1)(x+1) = x^2 + (x+2)(x+3)$
8. $3(x+12) - (x-4)^2 = 4 - x^2$
9. $(4x-1)^2 - (2x+3)^2 = (3x+2)(4x-5)$
10. $3(2x-7) - (2x-3)(2x-1) = 5 - (2x-5)^2$
11. $(3x-4)^2 - (3x-4)(2x+5) = 5(3x-4) + 3(x^2-12)$
12. $(15-8x)(15+8x) = 10 - (8x-5)^2$
13. $(3x-4)(3x+4) = 9x^2 + 6x - 4$
14. $2(x+2)(x-3) + 3x^2 = 5x^2 + 2$
15. $(2x+3)(2x-3) + 5x = 4x^2 + 1$

B

16. Find the number such that the square of 4 more than the number is the same as the square of the number.
17. Find two consecutive positive integers such that the difference of their squares is 51.
18. Find two consecutive odd positive integers such that the difference of their squares is 32.
19. Find two consecutive even positive integers such that the difference of their squares is 68.

7–15 Division of Polynomials

To be able to factor polynomials such as $x^3 - x^2 - 2x + 8$, it is helpful to be able to divide one polynomial by another. If you divide $x^3 - x^2 - 2x + 8$ by $x + 2$, for example, and get a polynomial for a quotient (it comes out "even"), then $x + 2$ and the quotient are factors of the original polynomial.

EXAMPLE 1. Divide $x^3 - x^2 - 2x + 8$ by $x + 2$ to see if $x + 2$ is a factor.

① Ask, By what must I multiply x in the divisor to get x^3 in the dividend? The answer is x^2. Write x^2 above x^3.

$$x+2 \overline{)\, x^3 - x^2 - 2x + 8}$$
$$x^2$$

② Multiply $x + 2$ by x^2, writing the answer below $x^3 - x^2 - 2x + 8$, like terms under one another. Then subtract as in ordinary division.

$$x+2 \overline{)\, x^3 - x^2 - 2x + 8}$$
$$x^2$$
$$x^2(x+2) \rightarrow \underline{x^3 + 2x^2}$$
$$-3x^2 - 2x + 8$$

③ Ask, By what must I multiply x in the divisor to get $-3x^2$? Write $-3x$ in the quotient.

$$x+2 \overline{)\, x^3 - x^2 - 2x + 8}$$
$$x^2 - 3x$$
$$\underline{x^3 + 2x^2}$$
$$-3x^2 - 2x + 8$$

④ Multiply $x + 2$ by $-3x$, and subtract.

$$-3x(x+2) \rightarrow \underline{-3x^2 - 6x}$$
$$4x + 8$$

(*Continued on next page.*)

(5) Ask, By what must I multiply x to get $4x$? Write 4 in the quotient, multiply $x+2$ by 4, and subtract.

$$
\begin{array}{r}
x^2-3x+4 \\
x+2\,\overline{)\,x^3-x^2-2x+8} \\
x^3+2x^2 \\
\hline
-3x^2-2x+8 \\
-3x^2-6x \\
\hline
+4x+8 \\
4x+8 \\
\hline
0
\end{array}
\qquad 4(x+2)
$$

Since the remainder is 0, $x+2$ is a factor. Also, x^2-3x+4 is a factor.

$$x^3-x^2-2x+8=(x+2)(x^2-3x+4)$$

Check the answer by multiplying.

EXAMPLE 2. Is $(3a^2-7ab+4b^2)\div(a-2b)$ a polynomial?

Divide.

$$
\begin{array}{r}
3a-b \\
a-2b\,\overline{)\,3a^2-7ab+4b^2} \\
3a^2-6ab \\
\hline
-ab+4b^2 \\
-ab+2b^2 \\
\hline
2b^2
\end{array}
\qquad
\begin{array}{l}
\\
\\
3a(a-2b) \\
\\
-b(a-2b)
\end{array}
$$

Since the remainder is not zero, the *quotient is not a polynomial.* Is the division correct? To check, the polynomial $3a^2-7ab+4b^2$ should be equal to the product of $a-2b$ and $3a-b$ plus the remainder, $2b^2$. Is the following true?

$$(a-2b)(3a-b)+2b^2 \stackrel{?}{=} 3a^2-7ab+4b^2$$
$$(3a^2-7ab+2b^2)+2b^2=3a^2-7ab+4b^2$$

Then the division is correct.

Exercises

A — Divide to see if the second polynomial is a factor of the first.

1. $c^2-3c-28,\ c-7$
2. $6x^2-28x+30,\ 2x-6$
3. $6x^3+x^2-19x+6,\ 2x-3$
4. $x^2-12x+40,\ x-6$
5. $a^3+6a^2-2a-12,\ a^2-2$
6. $6a^2-13a+7,\ 2a-3$
7. $x^2+xy-3y^2,\ x-y$
8. $p^2-p-32,\ p+5$
9. $3a^3+8a^2b+7ab^2+5b^3,\ 3a+2b$
10. $a^3-5a^2+2a+8,\ a-2$

B — Divide the first polynomial by the second.

Example. $(x^3+6x+20)\div(x+2)$

There is no x^2 term. But when you write the dividend, there must be a term for each power of the variable. Thus, write the x^2 term with 0 as the coefficient and divide as usual.

$$
\begin{array}{r}
x^2-2x+10 \\
x+2\,\overline{)\,x^3+0x^2+6x+20} \\
x^3+2x^2 \\
\hline
-2x^2+6x+20 \\
-2x^2-4x \\
\hline
10x+20 \\
10x+20 \\
\hline
0
\end{array}
$$

11. $x^3+4x-3,\ x+5$
12. $x^3-6x^2+1,\ x+2$
13. $x^3-2x-21,\ x-3$
14. $x^3-8,\ x-2$
15. $p^3-125,\ p-5$
16. $x^4+4x^2+16,\ x^2+2x+4$

More Challenging Problems

1. Find the square of $(a+b+c)$.

— Factor each polynomial.

2. $9a^4-12a^2b^2+4b^4$
3. $(x+y)^2+(x+y)-6$
4. $a^2-2ab+b^2-b^2+2bc-c^2$
5. $(x^2+6x+9)-(y-1)^2$
6. $(m^2+2mn+n^2)-(9x^2-6xy+y^2)$
7. $4a^2+4ab+b^2-9c^2+12cd-4d^2$
8. $x^2(2x-3)-2x(2x-3)+(2x-3)$

— One factor of each polynomial is written. Find the other factor by division.

9. $x^3+y^3=(x+y)(\underline{?})$
10. $x^3-y^3=(x-y)(\underline{?})$
11. $8a^3-27b^3=(2a-3b)(\underline{?})$
12. $x^5+1=(x+1)(\underline{?})$
13. $x^6-y^3=(x^2)^3-y^3=(x^2-y)(\underline{?})$

— Factor.

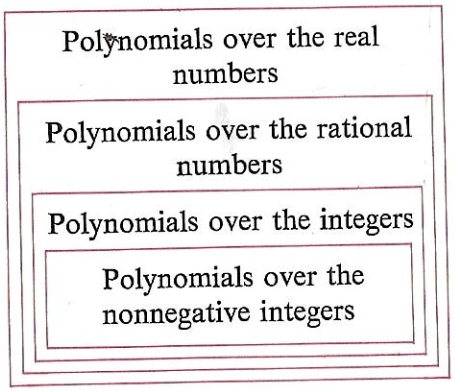
14. a^3+b^3
15. a^3-b^3
16. $125x^3-8y^3$
17. $64m^3+216n^3$
18. a^6-b^3
19. $64a^6+27b^3$
20. $m^2+n^2+p^2+2mn+2mp+2np$ (*Hint:* See Exercise 1.)
21. Find the greatest common factor of a^4-1, a^5+a^3, and a^6+1.

CHAPTER REVIEW

1. Know the meaning of and be able to use each of the following words or phrases. The number shown after each word or phrase indicates where it is introduced, in case you need to review.

polynomial in one variable (*308*)
term (*309*)
monomial (*309*)
binomial (*309*)
trinomial (*309*)
factor (*324*)
prime-factorization form (*324*)
unique factorization theorem (*325*)
common monomial factor (*326*)
prime polynomial (*326*)
perfect square trinomial (*330*)
binomial factor (*331*)

— The diagram shows the relationship of the various sets of polynomials. To which set does each of the following belong?


Polynomials over the real numbers
Polynomials over the rational numbers
Polynomials over the integers
Polynomials over the nonnegative integers

2. $5x^2 - 2x + 3$
3. $x^2 + 7x + 10$
4. $8x^2 - 5x + 3$
5. $7x^2 - \frac{2}{5}x + \frac{3}{4}$

6. Write an example of a **a.** monomial. **b.** binomial. **c.** trinomial.

— Write the additive inverse for each of the following.

7. $-3x^2 + 5$ **8.** $2x^2 - 4x + 3$ **9.** $\frac{x^2}{3} + \frac{2}{3}x - 6$

10. Add $3x^2 - 2x + 5$ and $-7x^2 + 4x - 2$.
11. Add $8x^3 - 2$ and $-5x^2 - 3x + 6$.
12. Subtract $9x^2 - 2x + 7$ from $-2x^2 + 6x - 1$.
13. From $-19x^2 + 3$, take $15x^2 - 2x + 7$.
14. Find the sum of $3x - 7$, $5x^2 + 6$, and $-2x^2 + 3x$.
15. Subtract $17x - 2$ from $5x^2 - 2x + 8$.
16. What postulate has been used here?

$$-3x(2x + 7) = -3x(2x) + (-3x)(7)$$

Complete the multiplication.

— Rename each product in polynomial form, by inspection if you can.

17. $(m^2 - 3)(m^2 + 4)$ **18.** $\left(\frac{n}{3} + 5\right)\left(\frac{2n}{3} - 4\right)$
19. $(7 - a)^2$ **20.** $(2x + 11)^2$
21. $(b - \frac{2}{3})(b + \frac{2}{3})$ **22.** $(3y^2 - 2)(3y^2 + 2)$

23. Why is $8x^2 - 3xy + y^2$ a prime polynomial? Why is $-18ab^2 + 13b$ not a prime polynomial?
24. Are the factors of a perfect square trinomial equal? Explain.

— Given the following combinations of signs for a trinomial, what will the signs be in each of the binomial factors?

25. +, +, + **26.** +, −, + **27.** +, +, − **28.** +, −, −

— Completely factor the following.

29. $4x^2 - y^2$ **30.** $x^2 + 3x + 2$
31. $10x^2 + 7x - 12$ **32.** $16x^2 + 8xy + y^2$

33. Divide $21x^2 + 25x - 4$ by $3x + 4$.

CHAPTER TEST

— Perform the operation indicated.

1. $(3x + 2)(5x - 9)$ **2.** $(8x - 3)(8x - 3)$
3. $(11x - 4)(11x + 4)$ **4.** $(5m + n)^2$
5. $2x(x^4 + 2x^3 + x^2 - 4x - 1)$
6. $(6r^3 - r + 10) - (3r^3 + 2r^2 - 5)$
7. $(2s^2 + st + 4t^2) + (5st - 6t^2 - s^2)$
8. $(x^2 + 3x - 54) \div (x + 9)$
9. $(6ab + 2bc) + (-3ac + 2ab) - (7bc - 2ac)$
10. $(x^3 + 3x^2 + 3x + 1) \div (x + 1)$

— Completely factor the following polynomials.

11. $3x^4 - 2x^3 - 5x^2$ **12.** $4x^2 - y^2$
13. $x^2 + 10x + 25$ **14.** $25x^2y^2 - 36z^2$
15. $a(b + c) + d(b + c)$ **16.** $x^2 - 11x + 24$
17. $x^2 - 12x + 35$ **18.** $8mn - 3 + 12n - 2m$

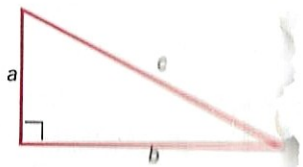
— Solve for x.

19. $(x + 3)(x - 5) = x^2 + 17x - 9$
20. $(2x + 1)^2 = 4(x^2 - 2x + 1)$

ENRICHMENT

Diophantine Solutions of $a^2 + b^2 = c^2$

The Pythagorean theorem states that the relationship among th sides of a right triangle is $a^2 + b^2 = c^2$. The triple of numbers $a =$ $b = 4$, and $c = 5$ satisfies this equation; that is, $3^2 + 4^2 = 5^2$ is true. Now one may assign any value to a and any value to b and then compute c. For example, if a is 1 and b is 2, then c is $\sqrt{5}$. And even though $\sqrt{5}$ is an irrational number, it is an acceptable solution.



The Greek mathematician Diophantus (about A.D. 250) was very interested in solution sets for algebraic equations, but he insisted that the solutions be natural numbers. Algebraic equations whose solutions are restricted to the natural numbers have come to be called Diophantine equations. Actually, at the time of Pythagoras, 750 years before Diophantus, formulas were known that would produce natural-number solutions to the Pythagorean theorem. Check the following formulas by substituting them in the Pythagorean theorem.

$$a = n^2 - 1 \qquad b = 2n \qquad c = n^2 + 1$$

You know from working with the Pythagorean theorem that if a is 5 and b is 12, then c will be 13. But you will notice that $a = n^2 - 1$ never produces $a = 5$ no matter what natural number is assigned to n. Hence, the above formulas given by Pythagoras did not produce all the possible natural-number triples that satisfy $a^2 + b^2 = c^2$.

Euclid (about 300 B.C.) proposed the following formulas, which can be shown to produce all possible Pythagorean triples of the Diophantine type. In the formulas, p, q, and r may be assigned any natural numbers, provided that q is greater than r.

$$a = p(q^2 - r^2) \qquad b = p(2qr) \qquad c = p(q^2 + r^2)$$

Check to see that the formulas satisfy the Pythagorean theorem. Then if q is 3, r is 2, and p is 1, $p(q^2 - r^2)$ is $1(9 - 4)$, or a is 5. Are b and c equal to 12 and 13, respectively?

Exercises

1. Use the Pythagorean formulas to produce values for a, b, and c for several choices of n.

2. Let $p = 2$, $q = 4$, and $r = 3$. Determine the corresponding Diophantine triple from Euclid's formulas.

3. Can you find values of p, q, and r in Euclid's formulas for Diophantine triples that give the same triples as some n in Pythagoras' formulas?

4. Since p, q, and r may be any natural number ($q > r$), can you see how to derive Pythagoras' equations for a, b, and c from Euclid's equations? (*Hint:* See what happens if you let p be 1 and r be 1.)

THE GROWTH OF MATHEMATICS

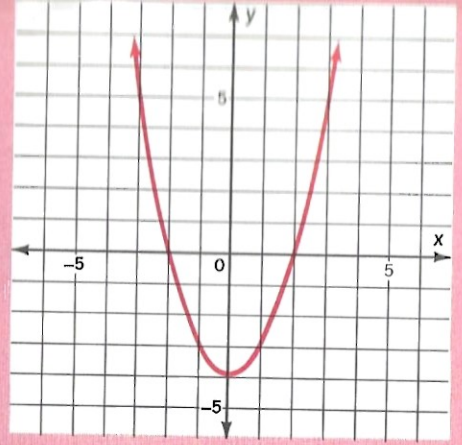
Mathematics in the Eastern World

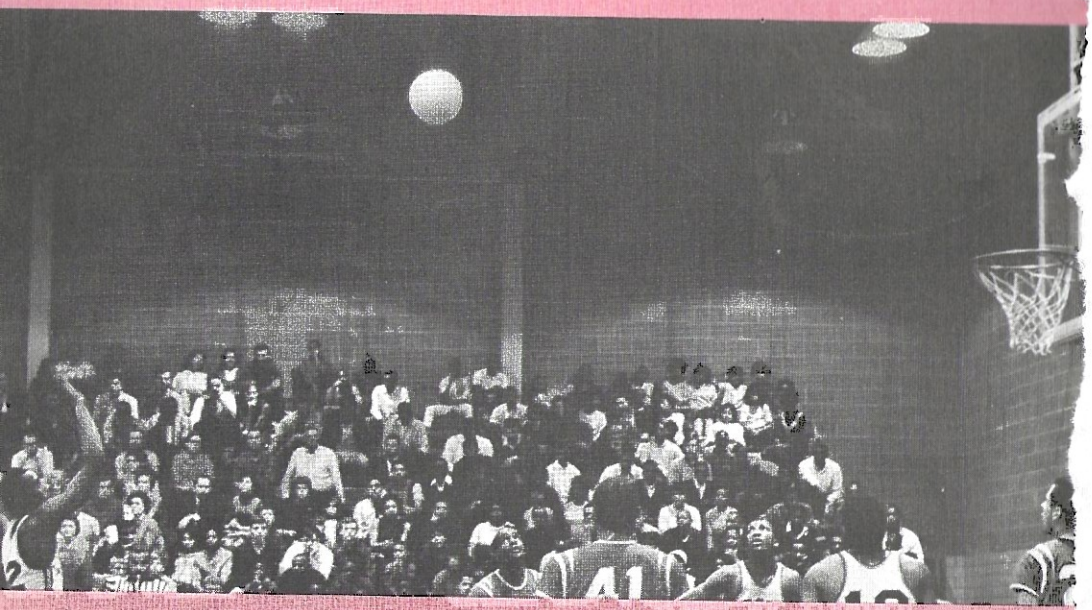
Although there is a long history of mathematical accomplishment in China, including a good system of numeration and a very early statement of the Pythagorean theorem, very little of Chinese thought reached the Western world directly. India, on the other hand, because of its geographical position was an important intermediary of ideas. It received ideas from the troops of Babylonians and Greeks who invaded the country, and the whole world benefited from Indian contributions to arithmetic and algebra carried to Europe by the Arabs.

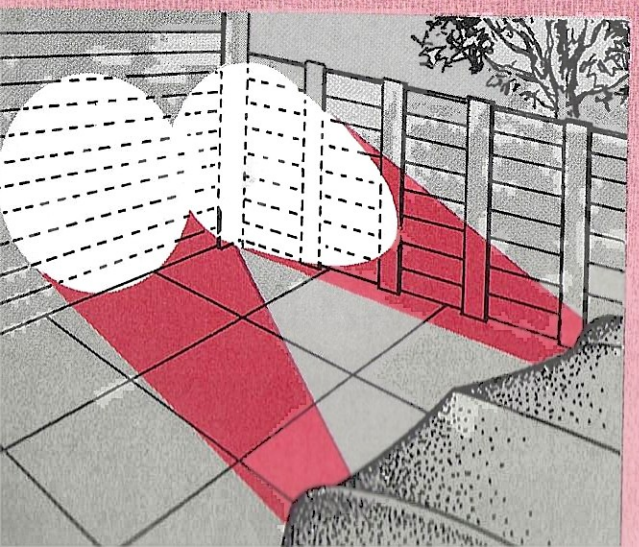
The most important contributions of the Indians occurred between the sixth and thirteenth centuries A.D. Early in that period, they began to use zero as a number, giving rules for addition, subtraction, multiplication, and division, with zero included. (At least one Indian mathematician made the common mistake of thinking division of a number by zero gives the original number.)

Unlike the Greeks, Indian mathematicians did not worry about proof or logical niceties. They just proceeded to calculate on the assumption that no problems would develop. As a consequence, Indian mathematicians were the first to use negative numbers with freedom. They made no distinction in practice between rational and irrational numbers, so they developed sound rules for calculating with radicals.

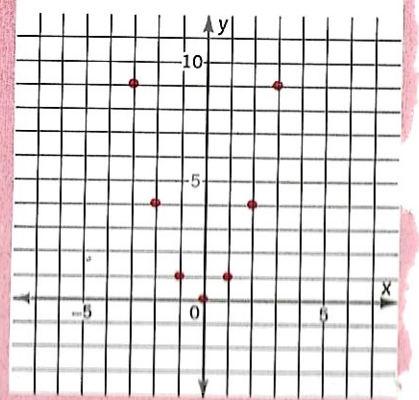
In algebra, the Indians knew Diophantus' work and went considerably beyond it. They could solve all equations involving squares of the variable, and they worked with Diophantine equations involving two variables. Much of their later work in mathematics was applied to astronomy, and it involved quite modern methods of calculation.







x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9



CHAPTER 8

QUADRATIC FUNCTIONS AND EQUATIONS

The curves pictured on the opposite page each illustrate one of the most interesting and useful of geometric curves, the parabola.

- The beam of a headlight is in the form of a cone. In the lower left drawing, the image of the car headlight on a wall perpendicular to the light is a circle. The image on the surface parallel to the light is a parabola.
- The path traveled by a basketball is a parabola. The boy throwing the ball in the center photograph is aiming for the hoop and has to determine a particular parabola if he wants the ball to go through the hoop. What is the path followed by a bullet or a rocket? What mathematics would you use to make a direct hit?
- If you completed the curve formed by the spray of water from the fountain, what curve would you have?
- The equation $y = x^2$ defines a quadratic function. The table above the graph at the lower right shows seven pairs of numbers that make $y = x^2$ true. How are the points arranged on the graph? Which number pair has a zero value for y? What is the x value for this pair?
- At the upper left is the graph for $y = x^2 - 4$ $(x \in \mathrm{R})$. The graph is in the form of what curve? What are the coordinates of the points at which the curve crosses the x axis? What are the coordinates of the lowest point? Can you tell what the highest point would be? What is the difference between this curve and the curve formed by the path of the basketball? What curve would result if you graphed $y = x^2 + 4$? How would you compare the two curves? How is the part of the curve on the right side of the y axis like the part of the curve on the left side of the y axis? How are they different?

8–1 Quadratic Functions and the Parabola

The following equations define quadratic functions.

a. $y = x^2$
b. $y = -3x^2$
c. $y = (\frac{1}{2}x - 3)^2$
d. $y = x^2 - 5x + 6$

Definition A quadratic function is the set of ordered pairs (x, y) such that $y = ax^2 + bx + c$ is true with $a \in R$, $b \in R$, $c \in R$, and $a \neq 0$.

Note that the definition above is designed to exclude the linear functions. Since a cannot be zero by definition, $y = 0x^2 - 5x + 6$ is a linear function and cannot be classified as a quadratic.

How can you be sure that **c** above is a quadratic function? Any equation of the form $y = ax^2 + bx + c$ is a **quadratic equation** and defines a quadratic function. By multiplying $(\frac{1}{2}x - 3)$ by $(\frac{1}{2}x - 3)$, you get the equation $y = \frac{1}{4}x^2 - 3x + 9$, a quadratic equation. What are a, b, and c in **a–d**?

A quadratic equation with all three terms is a **complete quadratic equation.** Quadratic equations of the form $y = ax^2$, $y = ax^2 + bx$, $y = ax^2 + c$ are **incomplete quadratic equations.**

Graph for $y = ax^2 + bx + c$

To graph a quadratic function, find ordered pairs that satisfy the defining equation and then locate the points whose coordinates are these ordered pairs.

EXAMPLE. Graph $y = x^2 - 2x - 3$, where the domain is R.

① For convenience, choose integers for x and find the corresponding number for y.

x	−2	−1	0	1	2	3	4
y	5	0	−3	−4	−3	0	5

Locate a point for each ordered pair. Then the graph, so far, consists of seven points, as shown in Figure 1.

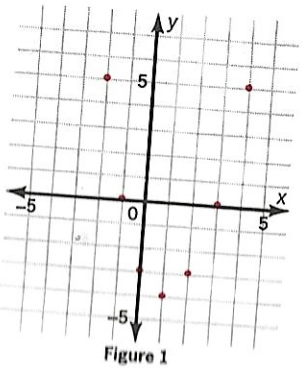


Figure 1

② If you choose rational numbers between the integers, you get many more points, some of which are shown in Figure 2.

③ If x is any real number, there will be no holes in the graph.

Sketch a smooth curve over the points to form the graph. The graph for a quadratic equation is a **parabola.** The arrows (Figure 3) show that the parabola continues upward on both sides.

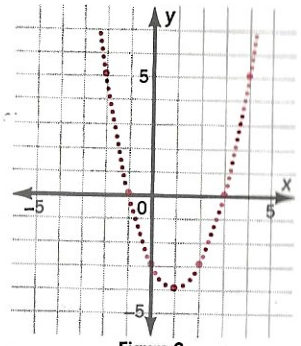


Figure 2

Examine Figure 3. What happens to y as x increases from 1 to 4? from 4 to 100? What happens to y as x increases from −100 to −2? from −2 to 1? Since y decreases on one side of the parabola and increases on the other, there must be a turning point. The turning point is easily seen on the graph as the lowest, or minimum, point, in this case, (1, −4). The turning point is called the **vertex** of the parabola.

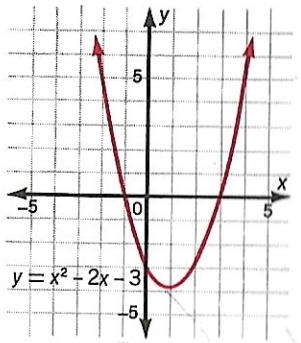
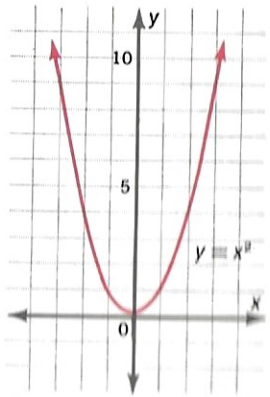


Figure 3

Parabola for $y = ax^2$

At the right is the graph for $y = x^2$, $x \in R$. Copy the graph on graph paper. On the same axes, draw the graph of $y = 2x^2$, $x \in R$. Now compare the graphs. Is the graph for $y = 2x^2$ wider or narrower than the one for $y = x^2$? On the same axes, draw the graph for $y = 3x^2$. How do they compare?



If $a > 1$, how does the graph for $y = ax^2$ compare with the graph for $y = x^2$?

Do you think the graph for $y = \frac{1}{2}x^2$ is outside or inside the graph for $y = x^2$? Draw it on the axes to be sure.

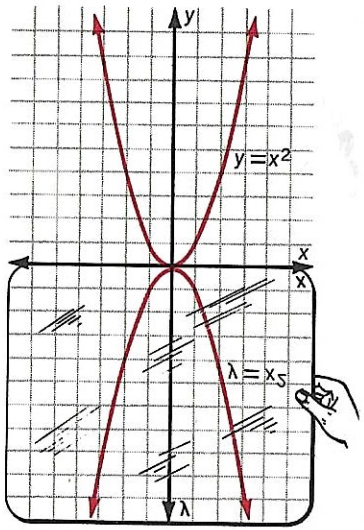
Be sure to label your graphs clearly and keep them to use for future exercises.

The table below shows corresponding numbers for x^2 and $-x^2$ for given values of x.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$-x^2$	-9	-4	-1	0	-1	-4	-9

From looking at the table, what would you guess is the graph for $y = -x^2$? How does it compare with that for $y = x^2$?

Graph $y = x^2$ and $y = -x^2$ on the same axes. If you placed a mirror on the x axis so that the graph of $y = x^2$ were reflected in it, what graph would the reflection be? One graph is the mirror image of the other.

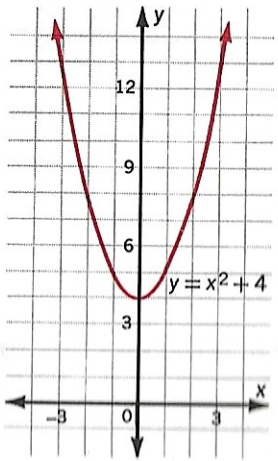


Parabola for $y = x^2 + c$

At the right is the graph of $y = x^2 + 4$. Copy it on graph paper. On the same axes, draw the graph of $y = x^2 - 4$. How do they compare? How do they both compare with $y = x^2$?

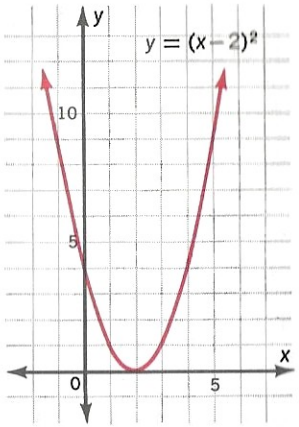
If $c > 0$, is the graph for $y = x^2 + c$ above or below that for $y = x^2$?

If $c < 0$, is the graph for $y = x^2 + c$ above or below that for $y = x^2$?



Parabola for $y = (x + a)^2$

At the right is the graph for $y = (x - 2)^2$, or $y = x^2 - 4x + 4$. Is it to the right or to the left of the graph for $y = x^2$? On graph paper, draw the graph for $y = (x + 2)^2$. How does it compare with the graph for $y = x^2$?



Now try these

Is each function defined below a quadratic function? Write *Yes* or *No*. If *Yes*, give the numbers for a, b, and c; if *No*, tell why not.

1. $y = x^2$
2. $y = 3x^2 - \sqrt{7}x + 6$
3. $y = 0x^2 - 7x + 9$
4. $y - 3x^2 = 0$
5. $6x^2 - 7x + 6 - y = 0$
6. $y = (x - 3)^2$
7. $y = 4(x - 3)^2$
8. $y = (x - 2)(x - 3)$
9. $y = x^3 + x^2 + 7x$
10. $y = (x - 2)(x + 2)(x - 1)$

Answers: **1.** Yes; $a = 1, b = 0, c = 0$ **2.** Yes; $a = 3, b = -\sqrt{7}, c = 6$ **3.** No; $a = 0$ **4.** Yes ($y = 3x^2$); $a = 3, b = 0, c = 0$ **5.** Yes ($y = 6x^2 - 7x + 6$); $a = 6, b = -7, c = 6$ **6.** Yes ($y = x^2 - 6x + 9$); $a = 1, b = -6, c = 9$ **7.** Yes ($y = 4x^2 - 24x + 36$); $a = 4, b = -24, c = 36$ **8.** Yes ($y = x^2 - 5x + 6$); $a = 1, b = -5, c = 6$ **9.** No; there is an x^3 term. **10.** No; $y = x^3 - x^2 - 4x + 4$ has an x^3 term.

Checkpoint

What is a quadratic function? How does it differ from a linear function? Describe the graph for a quadratic function.

Exercises

A **1.** Complete the table and graph $y = (x - 2)(x + 3)$ if the domain is the set of real numbers.

x	-5	-4	-3	-2	-1	0	1	2	3	4
y	14	6	0	?	?	?	?	?	?	?

— Graph the following ($x \in R$) by locating points using the x values given and then sketching a smooth curve.

2. $y = x^2 + 7$, $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
3. $y = x^2 - 6$, $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
4. $y = -x^2$, $\{-3\frac{1}{2}, -2\frac{1}{2}, -1\frac{1}{2}, 0, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}\}$
5. $y = (x + 3)^2$, $\{-7, -6, -5, -4, -3, -2, -1, 0, 1\}$
6. $y = -x^2 - 4$, $\{-3, -2, -1, 0, 1, 2, 3\}$
7. $y = x^2 - x - 12$, $\{-3, -2, -1, 0, \frac{1}{2}, 1, 2, 3, 4\}$

— Graph if the domain is R.

8. $y = 4x^2$
9. $y = -4x^2$
10. $y = x^2 - 5$
11. $y = x^2 + 5$
12. $y = (x - 1)(x + 1)$
13. $y = x^2 + 4x + 4$

B — Graph the following pairs of parabolas, and tell whether or not they intersect. If they do intersect, give the coordinates of the point or points of intersection.

14. $y = x^2 + 1$, $y = -x^2 + 1$
15. $y = (x + 2)^2$, $y = (x - 2)^2$
16. $y = (x - 3)^2$, $y = -x^2 + 9$
17. $y = 5 - x^2$, $y = -x^2$
18. $y = x^2 - 2$, $y = -x^2 + 6$

— Graph if the domain is R.

19. $y = 3(x - 1)(x + 2)$
20. $y = 2x^2 + 2x$
21. $y = 2x^2 - 7x - 4$
22. $y = x^2 - x - \frac{3}{4}$

C **23.** The function $y = (x - 1)(x + 2)(x - 3)$ is not quadratic. It is cubic.

a. Graph the function. Begin by using the following values for x.

$$x \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$$

Then choose some other rational values for x to get other points. Sketch a smooth curve that contains all the points.

b. As x increases above 3, what happens to the values for y? What happens to the curve?

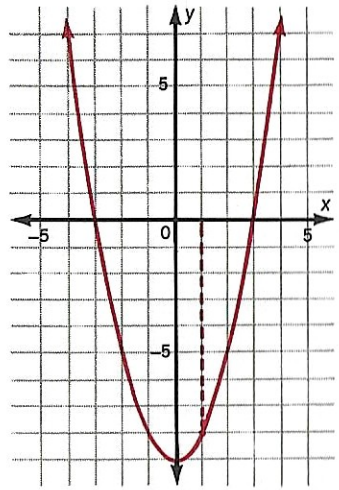
c. As x decreases below -2, what happens to y? What happens to the curve?

8–2 Zeros of a Function by Graphing

You have drawn the graph of a quadratic function by finding ordered pairs that satisfy the equation. Thus, you can also find ordered pairs of a function from looking at a graph.

EXAMPLE 1. Use the graph at the right to find the missing part of the following coordinates.

a. (1, ?) **b.** (0, ?)
c. (?, −5) **d.** (−1, ?)
e. (4, ?) **f.** (?, 0)



To find the second number in (1, ?), start at 1 on the x axis. Then move vertically (in this case, down along the line $x = 1$) until you hit the curve. Looking at the y axis, y is -8 at this point; so the coordinates are $(1, -8)$.

For (0, ?), simply move along the y axis ($x = 0$). The point on the curve is $(0, -9)$.

For (?, −5), start at -5 on the y axis, since this time you know the y coordinate. Move horizontally until you hit the curve. There are two points on the curve: $(2, -5)$ and $(-2, -5)$.

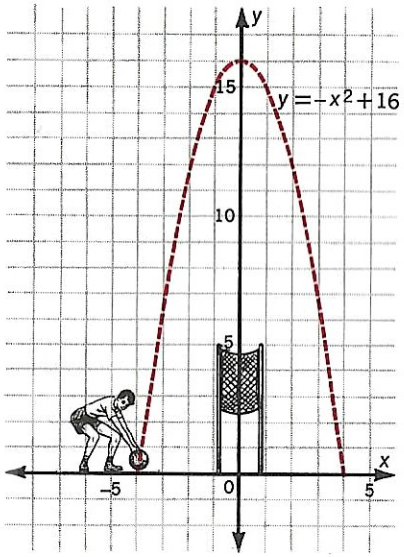
For (−1, ?), the y coordinate is -8.

For (4, ?), the y coordinate is 7.

For (?, 0), you move along the x axis because this is the line $y = 0$. There are two points on the curve: $(3, 0)$ and $(-3, 0)$.

The point $(0, -9)$ where the curve crosses the y axis is called the **y intercept**, as it was for the graph of a line crossing the y axis. Then $(3, 0)$ and $(-3, 0)$, the points where the curve crosses the x axis, are called **x intercepts.**

Suppose a volley ball is thrown from a point 4 feet to the left of a net, and it traces the arc of the parabola $y = -x^2 + 16$. The ball is 16 feet high when it crosses the net. How high was it at the beginning of the throw? How high will it be when it hits the floor? How does the graph show this?



Where will the ball hit the floor? It will hit 4 units to the right of the net. If you think of a line on the floor as the x axis and the net as the y axis, you can see from the graph that the ball started at $(-4, 0)$ and will hit the floor at $(4, 0)$.

Now think of the parabola for $y = -x^2 + 16$ continuing below the x axis. The points $(-4, 0)$ and $(4, 0)$ are the intercepts of the graph. What is the value of the function, $y = -x^2 + 16$, when the x numbers of the coordinates for the intercepts are substituted in the function?

$$\begin{aligned} y &= -(4)^2 + 16 \\ &= -16 + 16 \\ &= 0 \end{aligned} \qquad \begin{aligned} y &= -(-4)^2 + 16 \\ &= -(16) + 16 \\ &= 0 \end{aligned}$$

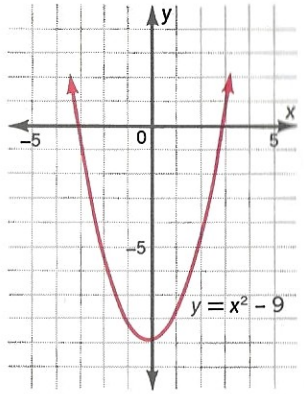
Finding the x number(s) that make the value of a function zero, $ax^2 + bx + c = 0$, is called finding the **zeros of a function.** As you saw in the example, this can be done by graphing to find the x intercepts.

Definition The zero of a function is a number for x in a set of ordered pairs of the form (x, y) such that y equals 0.

EXAMPLE 2. What are the zeros of $y = x^2 - 9$?

① Graph the function, as at the right.

② Find the zeros. The zeros are the numbers for x such that $0 = x^2 - 9$. From the graph, y is 0 where the curve crosses the x axis. The curve crosses at $(3, 0)$ and $(-3, 0)$. Hence, the zeros are 3 and -3.



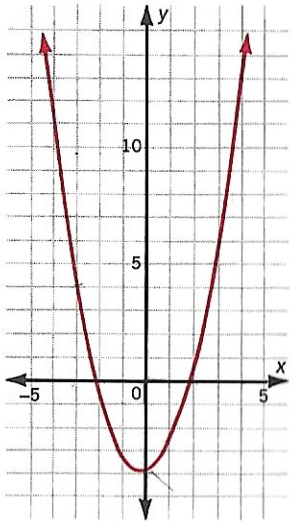
Check

$0 = 3^2 - 9$ $\qquad$ $0 = (-3)^2 - 9$

$0 = 9 - 9$ $\qquad$ $0 = 9 - 9$

Exercises

A (1.) Use the graph to find the missing coordinate. If there are two answers, give both.



a. $(0, \underline{?})$ $\qquad$ **b.** $(1, \underline{?})$

c. $(2, \underline{?})$ $\qquad$ **d.** $(3, \underline{?})$

e. $(-1, \underline{?})$ $\qquad$ **f.** $(-2, \underline{?})$

g. $(-3, \underline{?})$ $\qquad$ **h.** $(\underline{?}, 12)$

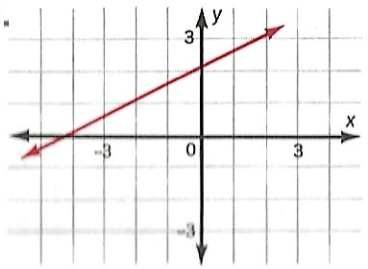
i. $(\underline{?}, 5)$ $\qquad$ **j.** $(\underline{?}, -3)$

k. $(\underline{?}, -4)$ $\qquad$ **l.** $(\underline{?}, 0)$

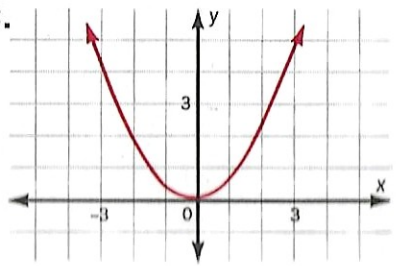
m. What are the zeros of the function graphed?

What values of x make y equal to zero for the functions graphed in Exercises 2–5? The functions are not all quadratic.

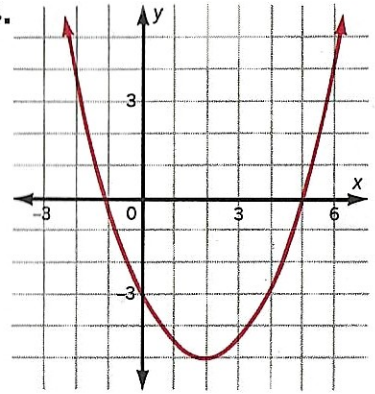
2.



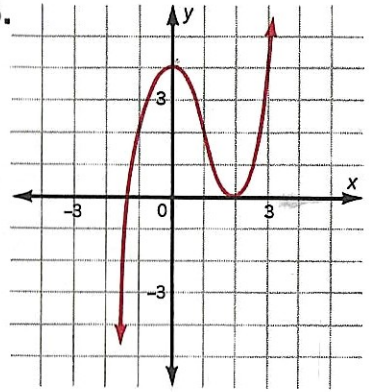
3.



4.



5.



Use your graphs from Section 8–1 to find the zeros of these functions. If the graph does not cross the x axis, there is no real number for x such that y is 0. In such a case, write *No real zeros.*

6. $y = x^2$ **7.** $y = x^2 + 4$ **8.** $y = (x - 2)^2$

9. $y = x^2 + 7$ **10.** $y = x^2 - 6$ **11.** $y = (x - 2)(x + 3)$

12. $y = -x^2$ **13.** $y = (x + 3)^2$ **14.** $y = (x - 2)(x + 2)$

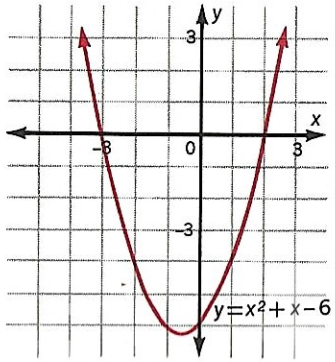
15. $y = 4x^2$ **16.** $y = -4x^2$ **17.** $y = (x - 4)(x + 3)$

B **18.** Solve $x^2 - 2x - 24 = 0$ (that is, find the values for x that make the equation true) by graphing $x^2 - 2x - 24 = y$.

19. Find the zeros of $y = 2x^2 - 3x - 2$ to the nearest half.

8–3 Solving Quadratic Equations by Factoring

At the right is the graph of the function $y = x^2 + x - 6$. The polynomial $x^2 + x - 6$ can be factored as $(x+3)(x-2)$. Thus, $y = (x+3)(x-2)$ is an equivalent equation for the function.



From the graph, you can see that $x = -3$ makes y equal to 0, and so does $x = 2$. This is easily checked algebraically.

$$\begin{aligned} y &= (x + 3)(x - 2) \\ &= (-3 + 3)(-3 - 2) \\ &= 0 \cdot (-5), \text{ or } 0 \end{aligned}$$

$$\begin{aligned} y &= (x + 3)(x - 2) \\ &= (2 + 3)(2 - 2) \\ &= 5 \cdot 0, \text{ or } 0 \end{aligned}$$

Solving $ax^2 + bx + c = 0$ $(a \neq 0)$ means to find the **solutions**, the number(s) for x that make the equation true. Putting the equation in factored form, you find the numbers that make $(x + c)(x + d) = 0$ true. You can do this by using Theorem 8–1.

Theorem 8–1 If the product of two real numbers, a and b, is zero, then $a = 0$ *or* $b = 0$ is true.

To find the solution set for $(x + c)(x + d) = 0$, use the compound sentence. It is helpful to have a symbol for *or*; $\vee$ is commonly used, and it is appropriate because it suggests $\cup$, the symbol for union.

$$\begin{gathered} (x + c = 0) \vee (x + d = 0) \\ (x = -c) \vee (x = -d) \end{gathered}$$

The solution set for $x + c = 0$ is $\{-c\}$, and the solution set for $x + d = 0$ is $\{-d\}$. The solution for $(x + c = 0) \vee (x + d = 0)$ is $\{-c\} \cup \{-d\}$, $\{-c, -d\}$. Does this check? First try $-c$, and then $-d$.

$$\begin{aligned} (x + c)(x + d) &= 0 \\ (-c + c)(-c + d) &= 0 \\ 0 \cdot (-c + d) &= 0 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} (x + c)(x + d) &= 0 \\ (-d + c)(-d + d) &= 0 \\ (-d + c) \cdot \quad 0 \quad &= 0 \\ 0 &= 0 \end{aligned}$$

They are both true by Theorem 2–4. Now you can solve quadratic equations by factoring.

EXAMPLE 1. Solve by factoring: $a^2 - 3a - 10 = 0$.

First factor the equation and then find the solution set.

$$\begin{gathered} a^2 - 3a - 10 = 0 \\ (a - 5)(a + 2) = 0 \\ (a - 5 = 0) \vee (a + 2 = 0) \\ (a = 5) \vee (a = -2) \end{gathered}$$

Check these solutions in the *original equation.*

Check $a = 5$.

$$\begin{aligned} 5^2 - (3 \cdot 5) - 10 &= 0 \\ 25 - 15 - 10 &= 0 \\ 25 - 25 &= 0 \\ 0 &= 0 \end{aligned}$$

Check $a = -2$.

$$\begin{aligned} (-2)^2 - (3 \cdot -2) - 10 &= 0 \\ 4 + 6 - 10 &= 0 \\ 10 - 10 &= 0 \\ 0 &= 0 \end{aligned}$$

Solution set: $\{5, -2\}$

Remember that if the equation is not in the form $ax^2 + bx + c = 0$, you must put it in that form to get the solution set.

EXAMPLE 2. Solve $x^2 + x + 9 = x + 14$.

First make the equation equal to 0. Add $-x - 14$ to both sides.

$$\begin{array}{rl} x^2 + x + 9 = & x + 14 \\ -x - 14 = & -x - 14 \\ \hline x^2 \quad - 5 = & 0 \end{array}$$

Then factor.

$$(x + \sqrt{5})(x - \sqrt{5}) = 0$$
$$(x + \sqrt{5} = 0) \vee (x - \sqrt{5} = 0)$$
$$(x = -\sqrt{5}) \vee (x = \sqrt{5})$$

Check $x = -\sqrt{5}$.

$$(-\sqrt{5})^2 + (-\sqrt{5}) + 9 = (-\sqrt{5}) + 14$$
$$5 + (-\sqrt{5}) + 9 = (-\sqrt{5}) + 14$$
$$(-\sqrt{5}) + 14 = (-\sqrt{5}) + 14$$

Check $x = \sqrt{5}$.

$$(\sqrt{5})^2 + \sqrt{5} + 9 = \sqrt{5} + 14$$
$$5 + \sqrt{5} + 9 = \sqrt{5} + 14$$
$$\sqrt{5} + 14 = \sqrt{5} + 14$$

Solution set: $\{-\sqrt{5}, \sqrt{5}\}$

You can see the advantage of algebraic methods. No graph would enable you to see that $-\sqrt{5}$ and $\sqrt{5}$ are zeros of $x^2 - 5 = y$.

Exercises

A Solve by factoring. Leave any irrational numbers in radical form.

1. $x^2 - 5x - 6 = 0$
2. $x^2 - 5x + 6 = 0$
3. $x^2 - 3x - 10 = 0$
4. $x^2 = 100$
5. $n^2 = 81$
6. $y^2 + 7y = -12$
7. $y^2 - 8y - 9 = 0$
8. $a^2 - 36 = 0$
9. $c^2 - 49 = 0$
10. $3x^2 = 75$
11. $y^2 = 4y + 12$
12. $y^2 = -7y - 10$
13. $5a^2 + 3a - 2 = 0$
14. $3a^2 - 10a - 8 = 0$
15. $6a^2 + a - 2 = 0$
16. $9x^2 = 36$
17. $n^2 = 7$
18. $y^2 = 8$

19. $12a^2 - 6a - 6 = 0$
20. $3n^2 + 7 = 28$
21. $15a^2 = 13a + 20$
22. $4a^2 - 9 = 11$
23. $3x^2 - 2 = x^2 + 6$
24. $x^2 - 7 = 5 - 3x^2$
25. $24a^2 + 26a - 63 = 0$
26. $5x^2 - 2x = 3$
27. $5a^2 + 15 = 9a^2 - 13$
28. $8 - n^2 = 7n^2 - 16$

B Find the solution set by factoring. Irrational solutions may be left in radical form.

Example. Solve $\frac{n^2}{2} - \frac{n}{3} = \frac{9 - n}{3}$.

First multiply both sides by 6, the least common multiple of the denominators, to get rid of the denominators, and then simplify the equation.

$$6\left(\frac{n^2}{2}\right) - 6\left(\frac{n}{3}\right) = 6\left(\frac{9 - n}{3}\right)$$
$$3n^2 - 2n = 2(9 - n)$$
$$3n^2 - 2n = 18 - 2n$$
$$3n^2 - 18 = 0$$

Now divide both sides by 3. Then factor and find the solution set.

$$\frac{3n^2}{3} - \frac{18}{3} = \frac{0}{3}$$
$$n^2 - 6 = 0$$
$$(n + \sqrt{6})(n - \sqrt{6}) = 0$$
$$(n + \sqrt{6} = 0) \vee (n - \sqrt{6} = 0)$$
$$(n = -\sqrt{6}) \vee (n = \sqrt{6})$$

Solution set: $\{-\sqrt{6}, \sqrt{6}\}$

29. $x^2 + \frac{x}{15} = \frac{x^2 + 2}{5}$
30. $\frac{5x^2 + 1}{2} - \frac{2x - 3}{7} = 2$
31. $\frac{n^2}{3} - \frac{n}{2} = \frac{4 - 2n}{4}$
32. $\frac{n^2}{5} + \frac{n}{3} = \frac{n + 3}{3}$
33. $\frac{n^2}{5} + \frac{n}{3} + \frac{1}{30} = \frac{2n + 5}{6}$
34. $\frac{2n^2 - 1}{3} = \frac{1}{2}$
35. $9x^2 = 2$
36. $4x^2 = 3$
37. $\frac{x^2}{4} + 5 = \frac{x^2 + 2}{2}$
38. $\frac{5}{2x} = \frac{x}{6}$
39. $\frac{3x + 5}{5} = \frac{9 + x}{x}$
40. $\frac{n^2}{3} + \frac{3 - n^2}{4} = 5$

C 41. Prove Theorem 8–1. (*Hint:* Assume that one factor is *not* zero, and solve the equation for the other factor.)

8–4 Solving Incomplete Quadratic Equations

An incomplete quadratic such as $x^2 - 9 = 0$ can be solved by factoring.

$$\begin{gathered} x^2 - 9 = 0 \\ (x + 3)(x - 3) = 0 \\ (x + 3 = 0) \vee (x - 3 = 0) \\ (x = -3) \vee (x = 3) \end{gathered}$$

Hence, $x^2 - 9 = 0$ is equivalent to $x = \pm 3$.

The example just given suggests that $x^2 = a$ is equivalent to $x = \pm\sqrt{a}$. It is easy to show that this is the case.

$$\begin{gathered} x^2 = a \\ x^2 - a = 0 \\ (x - \sqrt{a})(x + \sqrt{a}) = 0 \\ (x - \sqrt{a} = 0) \vee (x + \sqrt{a} = 0) \\ (x = \sqrt{a}) \vee (x = -\sqrt{a}) \\ x = \pm\sqrt{a} \end{gathered}$$

Theorem 8–2 Taking the square root of both sides of an equation produces an equivalent equation.

$\forall x \in R \quad \forall a \in R \quad x^2 = a$ **if and only if** $x = \pm\sqrt{a}$

This theorem may be used to solve $x^2 - 9 = 0$ in fewer steps. First rewrite as $x^2 = 9$. Take the positive and negative square root.

$$x = \pm 3$$

Solution set: $\{3, -3\}$

EXAMPLE 1. Solve $x^2 - 7 = 0$.

Add $+7$.	$x^2 = 7$
Take the square root.	$x = \pm\sqrt{7}$

Check $x = \sqrt{7}$.
$(\sqrt{7})^2 - 7 = 0$
$7 - 7 = 0$

Check $x = -\sqrt{7}$.
$(-\sqrt{7})^2 - 7 = 0$
$7 - 7 = 0$

Solution set: $\{\sqrt{7}, -\sqrt{7}\}$

An incomplete quadratic equation of the form $ax^2 + bx = 0$ can be solved by factoring.

EXAMPLE 2. Solve $2x^2 - 4x = 0$.

Factor.	$2x(x - 2) = 0$
Use Theorem 8–1.	$(2x = 0) \vee (x - 2 = 0)$
	$(x = 0) \vee (x = 2)$

The solution set is $\{0, 2\}$. Check the solutions in the original equation.

Now try these

Solve each equation.

1. $x^2 = 25$
2. $x^2 + x = 0$
3. $x^2 - 8 = 0$
4. $3x^2 = 15x$

Answers: 1. $x = \pm 5$ 2. $x = 0, x = -1$ 3. $x = \pm\sqrt{8}$, or $\pm 2\sqrt{2}$ 4. $x = 0, x = 5$

Exercises

A Give the solution set for each equation.

1. $x^2 = 16$
2. $y^2 - 4 = 0$
3. $l^2 - 11 = 0$
4. $n^2 - 7 = 12$
5. $m^2 + 2 = 3$
6. $7 - 2x^2 = -15$
7. $4c^2 = 36$
8. $3x^2 + 5 = 26$
9. $2r^2 = 200$
10. $2s^2 - 4s = 0$
11. $\frac{1}{2}z^2 = 32$
12. $x^2 + 7x = 0$
13. $x^2 - \frac{1}{2}x = 0$
14. $2x^2 = \frac{1}{2}$
15. $99 = b^2 - 22$
16. $3n^2 - n = 15 - n$
17. $4m^2 - 3 = 9$
18. $3y^2 + 5y = y^2$
19. $3z^2 = \frac{12}{9}$
20. $-2r - r^2 = 5r^2 + r$

B Solve for x; a, b, and c are constants.

21. $x^2 = a + b$
22. $\frac{x^2}{a} = 1$
23. $\frac{x^2}{a} = b$
24. $ax^2 - 2ax = 0$
25. $4x^2 = a^2$
26. $\frac{ax^2}{b} = 1$
27. $\frac{ax^2}{b} = ac$
28. $\frac{ab^2}{x^2} = 1$

8–5 Perfect Square Trinomials

Suppose you are asked to solve $x^2 - 6x + 7 = 0$. Simple factoring cannot be used to solve this equation (Try it!), and graphing does not yield exact answers. You might apply Theorem 8–2 to give either $\sqrt{x^2 - 6x + 7} = 0$ or $\sqrt{x^2 - 6x} = \pm\sqrt{7}$, but this does not help unless you know how to find $\sqrt{x^2 - 6x + 7}$ or $\sqrt{x^2 - 6x}$ — and you do not. There is a method of solving $x^2 - 6x + 7 = 0$, however, and it does depend upon Theorem 8–2. You begin by transforming one side of the equation into a *perfect square trinomial.* That part of the solution process is the subject of this section. The completion of the solution process is given in the next section.

Recall that a perfect square trinomial over the rational numbers is the square of a binomial over the rational numbers.

Binomial		Trinomial
$(x + 3)^2$	$= (x + 3)(x + 3)$	$= x^2 + 6x + 9$
$(r - 5)^2$	$= (r - 5)(r - 5)$	$= r^2 - 10r + 25$
$(y + b)^2$	$= (y + b)(y + b)$	$= y^2 + 2by + b^2$

The three binomials above consist of a variable and a constant. You can see that the first term of the trinomial is the square of the first term of the binomial.

$$x^2 \qquad r^2 \qquad y^2$$

The middle term is the variable with a coefficient. The coefficient of the middle term is twice the constant of the binomial.

$$6x = 2(3)x \qquad -10r = 2(-5)r \qquad 2by = 2(b)y$$

And the third term of the trinomial is the square of the second term of the binomial.

$$9 = (3)^2 \qquad 25 = (-5)^2 \qquad b^2 = (b)^2$$

Now suppose you are given an expression such as $x^2 - 6x$ (as in $x^2 - 6x = 7$). You should be able to supply a third term to $x^2 - 6x$ and make the expression a perfect square trinomial.

You know that the coefficient of the middle term is twice the constant of the binomial that would be squared to get the trinomial.

$$x^2 + 2(\underline{?})x + (\underline{?})^2 = (x + \underline{?})^2$$

In this case the coefficient of the middle term is -6.

$$-6 = 2(\underline{?})$$
$$-6 = 2(-3)$$

So the binomial is $(x - 3)^2$ and the last term of the trinomial would be $(-3)^2$, or 9.

$$(x - 3)^2 = x^2 + 2(-3)x + (-3)^2$$
$$= x^2 - 6x + 9$$

Then the constant of the binomial to be squared is one half the coefficient of the middle term of the perfect square trinomial, and the last term of a perfect square trinomial is the square of one half of the coefficient of the middle term, as symbolized here.

$$x^2 + bx + \left(\frac{b}{2}\right)^2$$

EXAMPLE. Complete $a^2 - 8a$ so it is a perfect square trinomial.

① The perfect square trinomial will be $a^2 - 8a + (\underline{?})^2$.

② The coefficient of the middle term is -8. Thus, $\frac{1}{2}(-8)$ is -4.

③ The trinomial is $a^2 - 8a + (-4)^2$, or $a^2 - 8a + 16$.

Now try these

— Supply the missing term to make each expression a perfect square trinomial.

1. $x^2 - 2x + 1$
2. $x^2 + 14x$
3. $y^2 + 3y$
4. $z^2 - \frac{1}{3}z$

Answers: 1. 1 2. 49 3. $\frac{9}{4}$ 4. $\frac{1}{36}$

Exercises

A — Find k.

1. $y^2 + (2 \cdot 7)y + k = (y + 7)^2$
2. $m^2 + (2 \cdot 9)m + k = (m + 9)^2$
3. $a^2 + 2(-5)a + k = (a - 5)^2$
4. $r^2 + (2 \cdot 9)r + k = (r + 9)^2$
5. $y^2 + 2(-\frac{1}{2})y + k = (y - \frac{1}{2})^2$
6. $x^2 + 2(-10)x + k = (x - 10)^2$
7. $x^2 + 6x + k = (x + 3)^2$
8. $y^2 + 10y + k = (y + 5)^2$
9. $x^2 - 12x + k = (x - 6)^2$
10. $m^2 - m + k = (m - \frac{1}{2})^2$

Supply the third term to make each of the following expressions a perfect square trinomial. Then give the binomial that would be squared to get the trinomial.

11. $x^2 + 10x$ **12.** $y^2 - 8y$ **13.** $m^2 + 4m$
14. $a^2 + 2a$ **15.** $r^2 - r$ **16.** $x^2 + 12x$
17. $r^2 - 20r$ **18.** $a^2 + a$ **19.** $x^2 + 5x$
20. $x^2 - 11x$ **21.** $x^2 - \frac{2}{5}x$ **22.** $y^2 + 7y$

For each of the following expressions, find the binomial that would be squared to give the expression as the first two terms of a perfect square trinomial.

23. $a^2 + 18a$ **24.** $y^2 - 22y$ **25.** $x^2 + 16x$
26. $x^2 - 3x$ **27.** $x^2 + \frac{4}{7}x$ **28.** $b^2 - 5b$
29. $x^2 + 8x$ **30.** $r^2 + \frac{1}{4}r$ **31.** $y^2 + \frac{1}{3}y$
32. $r^2 - 17r$ **33.** $a^2 + \frac{2}{3}a$ **34.** $x^2 - \frac{3}{4}x$

35. What must you add to $x^2 + ax$ to make it a perfect square trinomial? Then, using the addition property for equations, what would you add to both sides of $x^2 + ax = c$ to make the left side a perfect square trinomial?

What number must you add to both sides of the following equation to make the left side a perfect square trinomial?

36. $x^2 + 4x = 5$ **37.** $x^2 - 8x = -15$
38. $a^2 - 5a = 16$ **39.** $s^2 + 10s = 24$
40. $x^2 + \frac{3}{4}x = -7$ **41.** $x^2 - 14x = -9$
42. $x^2 + \frac{4}{7}x = \frac{3}{7}$ **43.** $a^2 + 7a = 18$

44. To apply the method of this section to the general quadratic equation $ax^2 + bx + c = 0$, you must rewrite the equation in the form $x^2 + \frac{b}{a}x = -\frac{c}{a}$, so that the coefficient of x^2 is 1 and the constant is on the right. To do this,

① add the additive inverse of the constant term c to both sides of the equation, and

② divide both sides by a.

Performing these two steps with the equation $5x^2 - 11x + 10 = 0$, you get the following.

$$5x^2 - 11x = -10$$
$$x^2 - \tfrac{11}{5}x = -\tfrac{10}{5}$$

Now try the two steps with the equation $3x^2 + 7x - 9 = 0$.

Change each of the following quadratic equations to the form $x^2 + \frac{b}{a}x = -\frac{c}{a}$, as in Exercise 44.

45. $7x^2 - 5x + 3 = 0$ **46.** $9x^2 - 6x = 4$
47. $8x^2 - 5x = 6$ **48.** $ax^2 = -bx - c$
49. $3x^2 + 4x = 5$ **50.** $6x^2 - 7x = -9$
51. $3x^2 + 5x + 1 = 0$ **52.** $7x^2 - 7x = 5$

8–6 Solving Quadratics by Completing the Square

The method of solving quadratic equations by **completing the square** involves making a perfect square trinomial of the left side of the equation and then using Theorem 8–2. First, consider equations in which the coefficient of x^2 is 1.

EXAMPLE 1. Solve $x^2 + 6x + 8 = 0$ by completing the square.

① Rewrite the equation to put on the left side only terms containing the variable.

$$x^2 + 6x + 8 - 8 = 0 - 8$$
$$x^2 + 6x = -8$$

② Make the left side of the equation a perfect square trinomial. Remember to add the same number to both sides of the equation.

$$x^2 + 6x + (\tfrac{6}{2})^2 = -8 + (\tfrac{6}{2})^2$$
$$x^2 + 6x + 9 = -8 + 9$$
$$x^2 + 6x + 9 = 1$$

③ Factor the trinomial into its two equal binomial factors.

$$(x + 3)^2 = 1$$

④ Apply Theorem 8–2.

$$x + 3 = \pm\sqrt{1}, \text{ or } \pm 1$$

⑤ Find the two solutions.

$$(x + 3 = 1) \vee (x + 3 = -1)$$
$$(x = 1 - 3) \vee (x = -1 - 3)$$
$$(x = -2) \vee (x = -4)$$

⑥ Check in the original equation.

$x = -2$
$(-2)^2 + 6(-2) + 8 = 0$
$4 - 12 + 8 = 0$
$-12 + 12 = 0$

$x = -4$
$(-4)^2 + 6(-4) + 8 = 0$
$16 - 24 + 8 = 0$
$-24 + 24 = 0$

Solution set: $\{-2, -4\}$

EXAMPLE 2. Solve $n^2 - 10 = 3n$.

① Rewrite the equation to put on the left side only terms containing the variable.

$$n^2 - 10 + 10 = 3n + 10$$
$$n^2 - 3n = 3n - 3n + 10$$
$$n^2 - 3n = 10$$

Follow steps ② through ⑥ as in Example 1. You should find that the solution set is $\{5, -2\}$.

Now try these

— Find the solution set by completing the square.

1. $x^2 + 4x = 21$
2. $x^2 - 2x = 15$
3. $x^2 - 9 = -8x$
4. $x^2 + 3x = \frac{7}{4}$

Answers: 1. $\{3, -7\}$ 2. $\{5, -3\}$ 3. $\{-9, 1\}$ 4. $\{\frac{1}{2}, -3\frac{1}{2}\}$

Exercises

A — Find the solution set by completing the square.

1. $x^2 + 4x = 12$
2. $x^2 + 6x = 27$
3. $n^2 - 8n = -15$
4. $n^2 - 2n - 8 = 0$
5. $a^2 - 6a + 8 = 0$
6. $x^2 + 8x = -12$
7. $a^2 + 4a = 5$
8. $b^2 + 10b = -16$
9. $x^2 - 14x = -40$
10. $x^2 - 12x = -35$
11. $x^2 - 10x = -24$
12. $c^2 + 2c = 15$
13. $x^2 - 16x = -60$
14. $x^2 - 16x = 17$
15. $a^2 - 3a = -2$
16. $x^2 - 7x + 12 = 0$
17. $x^2 - 7x + 10 = 0$
18. $x^2 - 9x = -18$
19. $a^2 - 3a = 10$
20. $n^2 + 6 = -5n$
21. $n^2 - 12 = n$
22. $x^2 - x = 6$
23. $x^2 + x = 6$
24. $x^2 - x = 30$
25. $y^2 + 3y = 4$
26. $y^2 - y = 2$
27. $p^2 + 5p = 0$
28. $x^2 = 3x$

B — Solve for x by completing the square.

29. $x^2 + 2ax = b^2$
30. $x^2 - ax = c$
31. $x^2 - x = k$
32. $x^2 - 2x - 4d = 0$
33. $x^2 - \frac{1}{h}x = 3k$
34. $x^2 + \sqrt{b}x - b^2 = 0$
35. $x^2 + 4e^2bx - 2f = 0$
36. $x^2 + 3ax = \frac{4b^2}{3}$

8–7 Extension of Completing the Square

All of the quadratic equations you have solved by the method of completing the square have had 1 as the coefficient of the squared term. To use this method in solving quadratics with any number as the coefficient of the squared term, one more step is needed, as you will see in the following example.

EXAMPLE. Solve $2x^2 - 9x - 5 = 0$.

Using the multiplication property for equations, multiply both sides of the equation by the reciprocal of the coefficient of the squared term.

①
$$2x^2 - 9x - 5 = 0$$
$$(\tfrac{1}{2} \cdot 2x^2) - (\tfrac{1}{2} \cdot 9x) - (\tfrac{1}{2} \cdot 5) = (\tfrac{1}{2} \cdot 0)$$
$$1x^2 - \tfrac{9}{2}x - \tfrac{5}{2} = 0$$

Now the coefficient of the squared term is 1; proceed as before.

②
$$x^2 - \tfrac{9}{2}x = \tfrac{5}{2}$$

③
$$x^2 - \tfrac{9}{2}x + (\tfrac{1}{2} \cdot -\tfrac{9}{2})^2 = \tfrac{5}{2} + (\tfrac{1}{2} \cdot -\tfrac{9}{2})^2$$
$$x^2 - \tfrac{9}{2}x + \tfrac{81}{16} = \tfrac{5}{2} + \tfrac{81}{16}, \text{ or } \tfrac{121}{16}$$

④
$$(x - \tfrac{9}{4})^2 = \tfrac{121}{16}$$

⑤
$$x - \tfrac{9}{4} = \pm\sqrt{\tfrac{121}{16}}, \text{ or } \pm\tfrac{11}{4}$$

⑥
$$(x - \tfrac{9}{4} = \tfrac{11}{4}) \vee (x - \tfrac{9}{4} = -\tfrac{11}{4})$$
$$(x = 5) \vee (x = -\tfrac{1}{2})$$

Solution set: $\{5, -\frac{1}{2}\}$

Check to see that the solutions are correct.

Exercises

A — Solve the following equations by completing the square. Irrational solutions may be left in radical form.

1. $3n^2 - 2n = 1$
2. $2n^2 = n + 3$
3. $5n^2 + 2 = 11n$
4. $3x^2 - 2x = 8$
5. $2y^2 - 3y = 35$
6. $4n^2 + 4 = 17n$
7. $6p^2 - 9p = 27$
8. $2x^2 - 7x + 3 = 0$
9. $3a^2 + 8a + 5 = 0$
10. $2a^2 - 5a - 12 = 0$
11. $3x^2 - x - 2 = 0$
12. $2p^2 - 2p - 1 = 0$
13. $2a^2 + 3a = 17$
14. $3 - 6x^2 = x$

15. $1 - y = 3y^2$ **16.** $8y^2 - y = 3$

17. $2x^2 = 9x + 3$ **18.** $2x^2 - x - 2 = 0$

19. $3b^2 + 5b - 1 = 0$ **20.** $10y^2 - 41y = -4$

B — Solve for x by completing the square.

21. $ax^2 + 2ax + a^2 = 0$ **22.** $2bx^2 + cx + 4d = 0$

23. $\sqrt{2}x^2 + ax = \frac{3c^2}{2}$ **24.** $g^2x^2 + \sqrt{h}x = -2l$

25. $\frac{1}{a}x^2 + \frac{1}{b}x + 2b = 0$ **26.** $ax^2 + bx + c = 0$

8–8 A Formula for Solving Quadratic Equations

By solving the general quadratic equation $ax^2 + bx + c = 0$ by the method of completing the square, you obtain two general solutions. The general solutions may then be used as formulas for finding the solutions of *any* quadratic equation.

Carefully examine the following solutions, one with numerals and the other with variables.

Solve $3x^2 - 16x + 5 = 0$.	Solve $ax^2 + bx + c = 0$.
① $x^2 - \frac{16}{3}x + \frac{5}{3} = 0$	$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
② $x^2 - \frac{16}{3}x = -\frac{5}{3}$	$x^2 + \frac{b}{a}x = -\frac{c}{a}$
③ $\left[\frac{1}{2}\left(-\frac{16}{3}\right) = -\frac{16}{6};\ \left(-\frac{16}{6}\right)^2 = \frac{16^2}{6^2}\right]$ $x^2 - \frac{16}{3}x + \frac{16^2}{6^2} = \frac{16^2}{6^2} - \frac{5}{3}$	$\left[\frac{1}{2}\left(\frac{b}{a}\right) = \frac{b}{2a};\ \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}\right]$ $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$
④ $\left(x - \frac{16}{6}\right)^2 = \frac{16^2 - 60}{6^2}$	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
⑤ $x - \frac{16}{6} = \pm\sqrt{\frac{16^2 - 60}{6^2}}$	$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4a}{4a^2}}$
⑥ $x = +\frac{16}{6} \pm \frac{\sqrt{16^2 - 60}}{6}$ $x = \frac{+16 \pm \sqrt{16^2 - 60}}{6}$	$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The solutions of the general quadratic equation are the following.

$$\left\{\frac{-b + \sqrt{b^2 - 4ac}}{2a},\ \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\}$$

By using the plus-or-minus sign, you have one **quadratic formula,** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, that may be used to find the solutions of *any* quadratic equation.

EXAMPLE 1. Solve $3x^2 - 16x + 5 = 0$ by using the quadratic formula.

First you must know what a, b, and c are to use the quadratic formula.

$$a = 3 \qquad b = -16 \qquad c = 5$$

Then use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(3)(5)}}{2(3)}$$

$$= \frac{16 \pm \sqrt{256 - 60}}{6}$$

$$= \frac{16 \pm \sqrt{196}}{6}, \text{ or } \frac{16 \pm 14}{6}$$

$$\left(x = \frac{16 + 14}{6}\right) \vee \left(x = \frac{16 - 14}{6}\right)$$

$$(x = 5) \vee (x = \tfrac{1}{3})$$

Check $x = 5$.	*Check* $x = \frac{1}{3}$.
$3(5)^2 - 16(5) + 5 = 0$	$3(\frac{1}{3})^2 - 16(\frac{1}{3}) + 5 = 0$
$75 - 80 + 5 = 0$	$\frac{3}{9} - \frac{16}{3} + 5 = 0$
$-80 + 80 = 0$	$-5\frac{1}{3} + 5\frac{1}{3} = 0$

Solution set: $\{5, \frac{1}{3}\}$

EXAMPLE 2. Use the quadratic formula to solve $3x^2 + 2x - 3 = 0$.

$$a = 3 \qquad b = 2 \qquad c = -3$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(3)(-3)}}{6}$$

$$x = \frac{-2 \pm \sqrt{4 + 36}}{6}, \text{ or } \frac{-2 \pm \sqrt{40}}{6}$$

Simplify the expression in the radical.

$$x = \frac{-2 \pm 2\sqrt{10}}{6}, \text{ or } -\frac{2}{6} \pm \frac{2}{6}\sqrt{10}, \text{ or } -\frac{1}{3} \pm \frac{1}{3}\sqrt{10}$$

The check is on the following page.

Check $x = -\frac{1}{3} + \frac{1}{3}\sqrt{10}$.

$$3(-\tfrac{1}{3} + \tfrac{1}{3}\sqrt{10})^2 + 2(-\tfrac{1}{3} + \tfrac{1}{3}\sqrt{10}) - 3 = 0$$
$$3[(-\tfrac{1}{3})^2 + 2(-\tfrac{1}{3} \cdot \tfrac{1}{3}\sqrt{10}) + (\tfrac{1}{3}\sqrt{10})^2] + 2(-\tfrac{1}{3} + \tfrac{1}{3}\sqrt{10}) - 3 = 0$$
$$3[\tfrac{1}{9} - \tfrac{2}{9}\sqrt{10} + \tfrac{1}{9}(10)] + (-\tfrac{2}{3} + \tfrac{2}{3}\sqrt{10}) - 3 = 0$$
$$\tfrac{3}{9} - \tfrac{6}{9}\sqrt{10} + \tfrac{30}{9} - \tfrac{2}{3} + \tfrac{2}{3}\sqrt{10} - 3 = 0$$
$$\tfrac{1}{3} + 3\tfrac{1}{3} - \tfrac{2}{3} - 3 - \tfrac{2}{3}\sqrt{10} + \tfrac{2}{3}\sqrt{10} = 0$$
$$+3\tfrac{2}{3} - 3\tfrac{2}{3} - \tfrac{2}{3}\sqrt{10} + \tfrac{2}{3}\sqrt{10} = 0$$

Check $x = -\frac{1}{3} - \frac{1}{3}\sqrt{10}$ to see that it is an exact solution.

Solution set: $\{-\frac{1}{3} + \frac{1}{3}\sqrt{10}, -\frac{1}{3} - \frac{1}{3}\sqrt{10}\}$

To find a *decimal approximation* for the solutions, you could look up $\sqrt{10}$ in the table on page 546 and compute the answer as follows.

$x \approx -\frac{1}{3} + \frac{1}{3}(3.162)$
$= -0.333 + 1.054$, or 0.721

$x \approx -\frac{1}{3} - \frac{1}{3}(3.162)$
$= -0.333 - 1.054$, or -1.387

Checking these values shows that they are only approximations.

Now try these

— Use the quadratic formula to solve the following equations.

1. $x^2 - 4x - 5 = 0$ **2.** $4x^2 + 12x = 7$ **3.** $3x^2 - 5x - 4 = 0$

Answers: **1.** $x = 5, x = -1$ **2.** $x = \frac{1}{2}, x = -\frac{7}{2}$ **3.** $x = \frac{5 \pm \sqrt{73}}{6}$

Exercises

A — Solve the following equations by the quadratic formula.

1. $x^2 - 4x = 21$
2. $x^2 + 6x = 16$
3. $2x^2 - x - 3 = 0$
4. $x^2 = 3x$
5. $p^2 - 4p = 0$
6. $n^2 - 2n - 8 = 0$
7. $a^2 + 4a = 5$
8. $6a^2 + 13a + 6 = 0$
9. $n^2 + 6 = -5n$
10. $2y^2 = 5y - 3$
11. $2x^2 + x = 3$
12. $y^2 + 3y = 4$
13. $y^2 - y = 2$
14. $10b^2 + 7b - 12 = 0$
15. $2a^2 + 11a + 5 = 0$
16. $x^2 + 6x = -9$

— Solve the following by the quadratic formula and give an approximation for the solutions to the nearest hundredth.

17. $3n^2 - 2n = 2$
18. $3x^2 - x - 3 = 0$
19. $2a^2 - 5a - 1 = 0$
20. $2n^2 = n + 2$
21. $5n^2 + 3 = 11n$
22. $3b^2 + 5b + 1 = 0$

— Solve the following by the quadratic formula. Simplify all answers. Irrational solutions may be left in radical form.

23. $x^2 - 2x = 4$
24. $x^2 + 2x = 1$
25. $x^2 + 4x = -2$
26. $x^2 + 6x + 3 = 0$
27. $x^2 - x - 1 = 0$
28. $x^2 + 3x + 1 = 0$
29. $2x^2 - 4x - 3 = 0$
30. $2x^2 + x - 5 = 0$
31. $3x^2 - 2x = 1$
32. $2x^2 + 7x + 2 = 0$
33. $6x^2 - 3x - 4 = 0$
34. $3x^2 + 10x + 5 = 0$

B — Use the quadratic formula to solve the following. Simplify all answers. Solve for x in Exercises 36, 41, and 46.

35. $2x^2 - 7x + 2 = 0$
36. $\frac{1}{a}x^2 + \frac{1}{b}x + \frac{1}{c} = 0$
37. $2x^2 - x - 1 = 0$
38. $3x^2 - 2x = 7$
39. $2y^2 - 3y = 4$
40. $4n^2 - 4 = n$
41. $\frac{1}{c}x^2 + dx = 2c$
42. $7x^2 - 3x - 1 = 0$
43. $9n^2 + 2 + 9n = 0$
44. $8n^2 + 7n - 2 = 0$
45. $4a^2 - 5 = 3a$
46. $3gx^2 - 2hx + k = 0$

8–9 Problems Involving Quadratic Equations

The conditions of the following exercises may be stated in terms of a quadratic equation, which you may then solve by any of the methods you have learned. Study the following Examples.

EXAMPLE 1. The altitude of a triangle is 3 units larger than its base. *The area is 9 square units.* Find the base and the altitude.

① Let x = number of units in the base.
Then $x + 3$ = number of units in the altitude.
Given 9 = number of square units in the area.
(Note that the replacement set for x is the set of positive real numbers.)

② Use these expressions in the formula for finding the area of a triangle.

$$\tfrac{1}{2}ba = A$$
$$\tfrac{1}{2}(x)(x + 3) = 9$$

(Continued on next page.)

③ Rewrite the equation in quadratic form.

$$\tfrac{1}{2}(x^2 + 3x) = 9$$
$$x^2 + 3x = 18$$
$$x^2 + 3x - 18 = 0$$

④ Solve the quadratic equation. (Factoring is used here.)

$$(x - 3)(x + 6) = 0$$
$$(x - 3 = 0) \vee (x + 6 = 0)$$
$$(x = 3) \vee (x = -6)$$

⑤ Since x, the measure of the base, is a member of the set of positive real numbers, -6 is not in the solution set. Does 3 satisfy the conditions of the problem?

$$x = 3$$
$$x + 3 = 6$$
$$\tfrac{1}{2}(3)(6) = 9$$
$$\tfrac{1}{2}(18) = 9$$

Solution set: $\{3\}$

⑥ The base is 3 units long and the altitude is 6 units long.

EXAMPLE 2. The sum of two numbers is 12. *The sum of their squares is 80.* Find each number.

① Let n = one number.
Then $12 - n$ = the other number.

② Use the conditions of the problem to write an equation.

$$n^2 + (12 - n)^2 = 80$$
$$n^2 + 144 - 24n + n^2 = 80$$
$$2n^2 - 24n + 64 = 0$$
$$n^2 - 12n + 32 = 0$$

③ Solve the equation.

$$(n - 4)(n - 8) = 0$$
$$(n = 4) \vee (n = 8)$$

④

Check $n = 4$.	*Check* $n = 8$.
$4^2 + 8^2 = 80$	$8^2 + 4^2 = 80$
$16 + 64 = 80$	$64 + 16 = 80$

Solution set: $\{4, 8\}$

⑤ Since $12 - 4 = 8$ and $12 - 8 = 4$, the two numbers are 4 and 8.

Exercises

A — Write a quadratic equation for each problem and solve. Draw a figure to help you if appropriate.

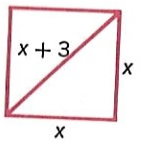
1. One number is 3 units larger than another. Their product is 54. Find the numbers.

2. The square of a number is 56 more than the number itself. What is the number?

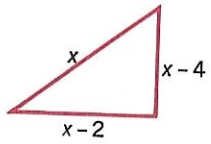
3. A number exceeds its square by $\frac{2}{9}$. Find the number.

4. The perimeter of a rectangle is 30 inches and its area is 54 square inches. What are its dimensions?

5. What is the length of a diagonal of a square if it is 3 inches longer than a side? (Use the Pythagorean theorem.) Approximate the answer to hundredths.



6. A man wished to fit three rods together in the shape of a right triangle. The hypotenuse was to be 2 inches longer than the base and 4 inches longer than the altitude. How long should he make each rod? Use the Pythagorean theorem.



$$(x - 2)^2 + (x - 4)^2 = x^2$$

7. The length of a rectangular piece of sheet metal is 3 times its width. What is its length if its area is 192 square inches?

8. The area of the front of the fireplace in the Wilson's living room is to be 1080 square inches. Its width is to be 6 inches longer than its height. What are its dimensions to be?

9. The product of two consecutive odd integers is 2703. What are the integers? (*Hint:* You can use $2n + 1$ and $2n + 3$. You should get a positive number and a negative number for the solutions of your equation. A pair of positive numbers and a pair of negative numbers satisfy the conditions of the problem.)

10. Find two consecutive even integers whose product is 1848. (*Hint:* You can use $2n$ and $2n + 2$.)

11. The difference in the squares of two consecutive odd integers is 56. What are the integers?

12. If a number is decreased by 18 times its reciprocal, the difference will be 7. What is the number?

13. The sum of a number and 15 times its reciprocal is 8. What is the number?

14. Two square playing spaces are marked off on a playground. The side of one playing space is 6 feet less than a side of the other playing space. The area of the two playing spaces is 1076 square feet. What is the length of a side of each playing space?

15. In the auditorium of Sally Bates's school, the number of seats in each row is 8 fewer than the number of rows. How many seats are in each row if the auditorium seats 609 persons?

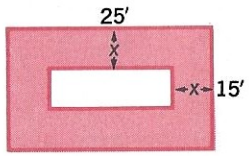
16. There are two numbers such that one is 3 more than $\frac{1}{2}$ the other. The difference of the squares of the two numbers is 495. What are the numbers?

B — Write a quadratic equation for each of the following problems and solve.

17. The numerator of a fraction is 1 less than its denominator. If the fraction is increased by 2 times its reciprocal, the sum will be $3\frac{5}{12}$. Find the numerator and the denominator.

18. The denominator of a fraction is 2 more than its numerator. If the fraction is increased by 3 times its reciprocal, the sum will be $5\frac{3}{5}$. Find the numerator and the denominator.

19. The area of the border in the figure at the right is 225 square feet. How wide is this border if the outside length is 25 feet and the outside width is 15 feet?



20. Sam and his father drove 294 miles to the State Fair. Sam drove the farm truck at an average speed of 7 miles per hour slower than his father drove the family car. It took Sam an hour longer than it did his father. What was Sam's rate? his father's rate?

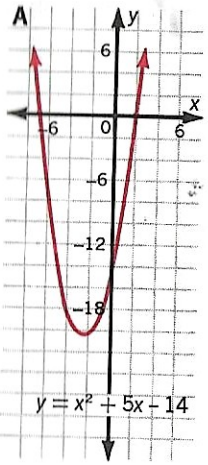
21. An airplane flew a round-trip training flight from airport A to airport B. The distance between the two airports was 1200 miles. Going against the wind, the pilot flew 60 miles per hour slower than returning. It took one hour more time going than returning. What was the speed going and returning?

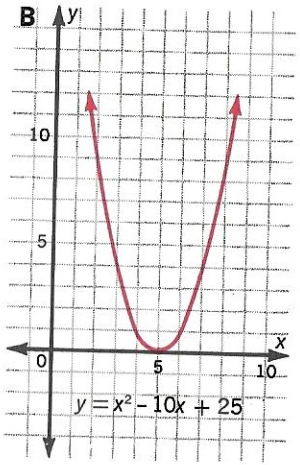
22. Mr. Fritz and Mr. Murphy can do a piece of work together in 5 hours. Alone Mr. Murphy can do the job in 3 hours less time than Mr. Fritz can do it alone. How many hours, to the nearest tenth, will it take each to do the job?

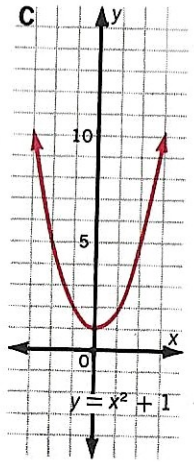
8–10 The Nature of the Solutions

Knowing the **nature of the solutions** of a quadratic equation is knowing if they are real or not and if there are two or only one.

From the graphs of quadratic functions, you can tell the nature of the solutions, or zeros, of the corresponding quadratic equations.







How many times does each graph cross the x axis in A? in B? in C? Then how many distinct real numbers make $x^2 + 5x - 14 = 0$ true? $x^2 - 10x + 25 = 0$ true? $x^2 + 1 = 0$ true?

For convenience, the solutions of $x^2 - 10x + 25 = 0$ are often said to be equal, because solving $x^2 - 10x + 25 = 0$ gives $+5$ and $+5$ as the solutions. However, there is only one *distinct* solution, $+5$.

Summary

Equation	Nature of solutions
$x^2 + 5x - 14 = 0$	Two distinct, real solutions.
$x^2 - 10x + 25 = 0$	One distinct, real solution (or two equal, real solutions).
$x^2 + 1 = 0$	No real solutions.

Using the quadratic formula yields the same results without graphing.

$$x^2 + 5x - 14 = 0 \qquad x = \frac{-5 \pm \sqrt{25 + 56}}{2}$$

Since $\sqrt{25 + 56}$ is *not equal* to zero, there are two real solutions, $\frac{-5 + \sqrt{81}}{2}$ and $\frac{-5 - \sqrt{81}}{2}$.

$x^2 - 10x + 25 = 0 \qquad x = \dfrac{10 \pm \sqrt{100 - 100}}{2}$

Since $\sqrt{100 - 100}$ is *equal* to zero, there is one distinct real solution, 5.

$x^2 + 1 = 0 \qquad x = \dfrac{0 \pm \sqrt{0 - 4}}{2}$

No real number equals $\sqrt{-4}$. You might try -2, but $(-2)^2$ is 4, not -4. Since $\sqrt{-4}$ is not a real number, $\dfrac{\pm\sqrt{-4}}{2}$ is not a real number, and there are no real solutions.

Can you see which portion of the quadratic formula determines the nature of the solutions? It is $b^2 - 4ac$. This number is the **discriminant** of the equation $ax^2 + bx + c = 0$. Why?

If $b^2 - 4ac > 0$, there are two distinct, real solutions for $ax^2 + bx + c = 0$.
If $b^2 - 4ac = 0$, there is one distinct, real solution (two equal, real solutions).
If $b^2 - 4ac < 0$, there are no real solutions.

Checkpoint

How can you use $b^2 - 4ac$ to determine the nature of the solutions of a quadratic equation?

Now try these

Determine the nature of the solutions.

1. $2x^2 - 3x - 1 = 0$
2. $n^2 - 2n + 4 = 0$
3. $x^2 - 6x + 9 = 0$
4. $x^2 - 5x + 6 = 0$

Answers: 1. Two distinct, real solutions. 2. No real solutions. 3. One distinct, real solution. 4. Two distinct, real solutions.

Exercises

A Without solving, give the nature of the solutions of each of the following equations.

1. $x^2 + 2x + 8 = 0$
2. $3x^2 - 3x - 4 = 0$
3. $3y^2 - \frac{1}{3}y = \frac{2}{5}$
4. $x^2 = 5x + 5$
5. $3x^2 + 4x - \frac{8}{3} = 0$
6. $2y^2 - 4y = -2$
7. $x^2 - x + \frac{6}{25} = 0$
8. $20x = x^2 + 100$
9. $\frac{1}{3}x^2 - 16x + 132 = 0$
10. $4x^2 + 2x - 9 = 0$

B
11. For what values of k will the graph of $x^2 + kx + 25 = y$ touch the x axis at just one point?
12. For what values of k are the solutions of $x^2 + 4x + k = 0$ real?
13. For what values of k will the graph of $x^2 + kx + 3 = y$ cut the x axis twice?
14. For what values of k will the graph of $kx^2 - 6x + 1 = y$ neither touch nor cross the x axis?

CHAPTER REVIEW

1. Know the meaning of and be able to use each of the following words or phrases. The number shown after each word or phrase indicates where it is introduced, in case you need to review.

quadratic function (*350*)
quadratic equation (*350*)
complete quadratic equation (*350*)
incomplete quadratic equation (*350*)
parabola (*351*)
vertex of a parabola (*351*)
y intercept (*355*)
x intercept (*355*)
zero of a function (*356*)
solution of a quadratic equation (*359*)
completing the square (*367*)
quadratic formula (*371*)
nature of the solution(s) of a quadratic equation (*377*)
discriminant (*378*)

The general form of the quadratic equation is $y = ax^2 + bx + c$. What is the value of a, b, and c in each of the following?

2. $y = -3x^2$
3. $y = 2x^2 + 5$
4. $y = 5x^2 - 2x + 7$
5. $y = -\frac{1}{4}x^2 + 9x$

6. Write an example of a complete quadratic equation; of an incomplete quadratic equation.

Describe the parabola that is the graph of each of the following; that is, tell which way it opens and give its vertex and x intercepts.

7. $y = 2x^2 - x - 15$
8. $y = 2x^2 - 3x$
9. $y = -5x^2$
10. $y = x^2$

What are the zeros of the following functions?

11. $3x^2 + 16x - 12 = 0$
12. $y = \frac{3x^2}{5}$

— Solve each equation by factoring.

13. $x^2 - 2x - 35 = 0$ **14.** $2x^2 - x - 21 = 0$

15. If you are given the equation $(x - 5)(x + 2) = 0$, why does replacing x by 5 make the equation true?

16. What are the x intercepts of the graph of $y = 9x^2 - 29x + 6$?

— Make each of the following expressions a perfect square trinomial.

17. $x^2 - 9x$ **18.** $x^2 - \frac{2}{5}x$ **19.** $x^2 + 5x$

— Solve the following equations by completing the square.

20. $x^2 - 8x - 3 = 0$ **21.** $x^2 - 5x = 4$

22. $15x^2 + 2x - 1 = 0$ **23.** $7x^2 - 17x + 7 = 0$

24. Write the general form of a quadratic equation. Write the quadratic formula. Explain how the two are related.

— Use the quadratic formula to solve the following equations.

25. $4x^2 + 5 = 9x$ **26.** $15x^2 - 7x - 2 = 0$

27. $2x^2 - 7x + 1 = 0$ **28.** $x^2 - 11x = 0$

29. "The sum of a number, x, and 14 times its reciprocal is 9." You could use $x + \frac{14}{x} = 9$ to express the sum. Write the equation as a quadratic equation and find the number or numbers that satisfy it.

— Use the discriminant to give the nature of the solutions of the following.

30. $15x^2 - 16x - 15 = 0$ **31.** $x^2 - 8x = -16$

32. $2x^2 - 3x + 3 = -11$ **33.** $2x^2 = 32$

CHAPTER TEST

— Make the following expressions perfect square trinomials by completing the square.

1. $x^2 + 6x$ **2.** $x^2 - 12x$ **3.** $n^2 - 2n$ **4.** $r^2 + 7r$

— Solve the following quadratic equations.

5. $x^2 + 5x + 6 = 0$ **6.** $a^2 - 6a - 7 = 0$

7. $3s^2 - 2s - 8 = 0$ **8.** $\frac{1}{2}t^2 + \frac{1}{2}t - 15 = 0$

9. $x^2 - 5 = 0$ **10.** $m^2 + 3m = 0$

11. $x^2 + 8x + 16 = 0$ **12.** $y^2 - 14y = 1$

13. $2r^2 + r - 6 = 0$ **14.** $4b^2 - 16 = 0$

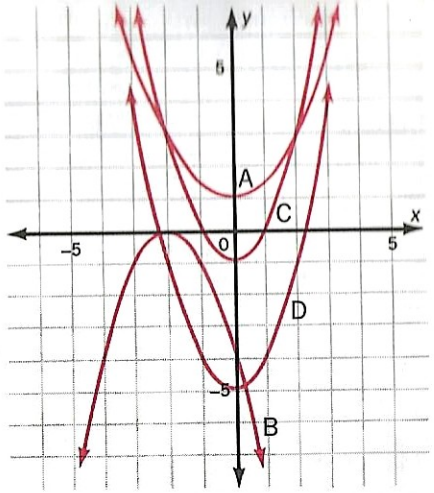
15. For each of the following equations, choose the correct graph at the right.

a. $y = -x^2 - 4x - 4$

b. $y = x^2 - 1$

c. $y = \frac{1}{2}x^2 + 1$

d. $y = x^2 - 5$



— Graph each of the following parabolas, and tell whether the vertex is a minimum or a maximum point.

16. $y = -x^2 + 2$ **17.** $y = x^2 + 10x + 25$

— Give the y intercept and the real zeros (if there are any) of the graphs of the following equations.

18. $3x^2 - 2x - 8 = y$ **19.** $-x^2 + 8x - 15 = y$

20. $x^2 + 4 = y$ **21.** $x^2 + 4x + 4 = y$

22. The length of a rectangle is 1 inch more than twice the width and the area is 55 square inches. Find the length and the width.

23. Separate 18 into two parts so that the sum of the squares of the parts will be 170.

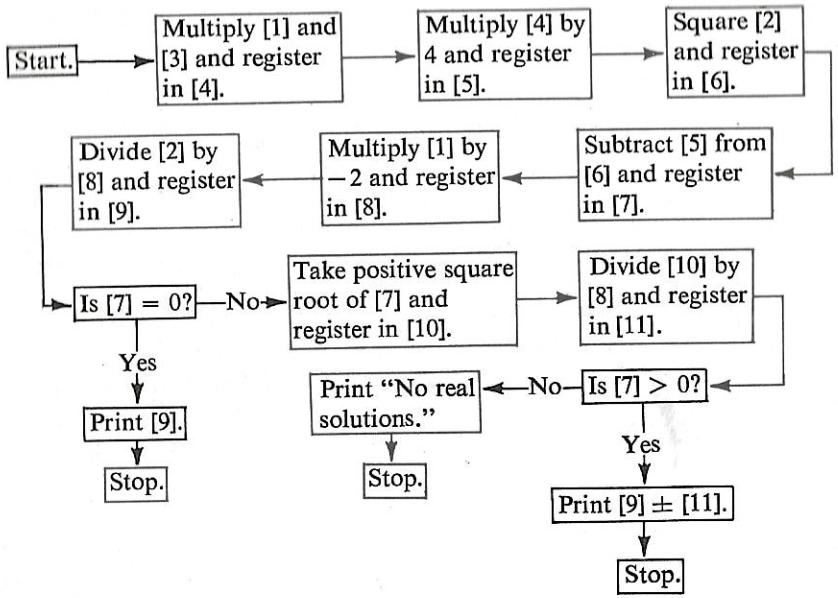
ENRICHMENT

A Computer Flow Chart for Solving Quadratic Equations

In Chapter 5 there is a flow-chart analysis of the solution of the linear equation. Now you will work with a flow chart for the solution of the quadratic equation. Remember that the computer can add, subtract, multiply, divide, take square roots, give absolute values, and compare given numbers with each other or with given constants. These and only these operations may be included in the flow chart.

Submit $ax^2 + bx + c = 0$, $a \neq 0$, $b \neq 0$, $c \neq 0$, to the computer with a in location [1], b in location [2], and c in location [3]. Read through the flow chart and follow the steps for the algebraic solution in the left column below the chart. Then follow the numerical solution in the right column.

Flow Chart for Solving Quadratic Equations


Start.
Multiply [1] and [3] and register in [4].
Multiply [4] by 4 and register in [5].
Square [2] and register in [6].
Subtract [5] from [6] and register in [7].
Multiply [1] by −2 and register in [8].
Divide [2] by [8] and register in [9].
Is [7] = 0?
No
Yes
Print [9].
Stop.
Take positive square root of [7] and register in [10].
Divide [10] by [8] and register in [11].
Is [7] > 0?
No
Print "No real solutions."
Stop.
Yes
Print [9] ± [11].
Stop.

Solve $ax^2 + bx + c$.	Location	Solve $3x^2 + x + 4$
a	[1]	3
b	[2]	1
c	[3]	−4
ac	[4]	−12
$4ac$	[5]	−48
b^2	[6]	1
$b^2 - 4ac$	[7]	49
$-2a$	[8]	−6
$\frac{b}{-2a}$	[9]	$-\frac{1}{6}$
$\sqrt{b^2 - 4ac}$	[10]	7
$\frac{\sqrt{b^2 - 4ac}}{-2a}$	[11]	$-\frac{7}{6}$
[7] > 0 [7] = 0 [7] < 0		
$\frac{b}{-2a} \pm \frac{\sqrt{b^2 - 4ac}}{-2a}$ $\frac{b}{-2a}$ "No real solutions."	Print result.	$-\frac{1}{6} \pm -\frac{7}{6}$

Exercises

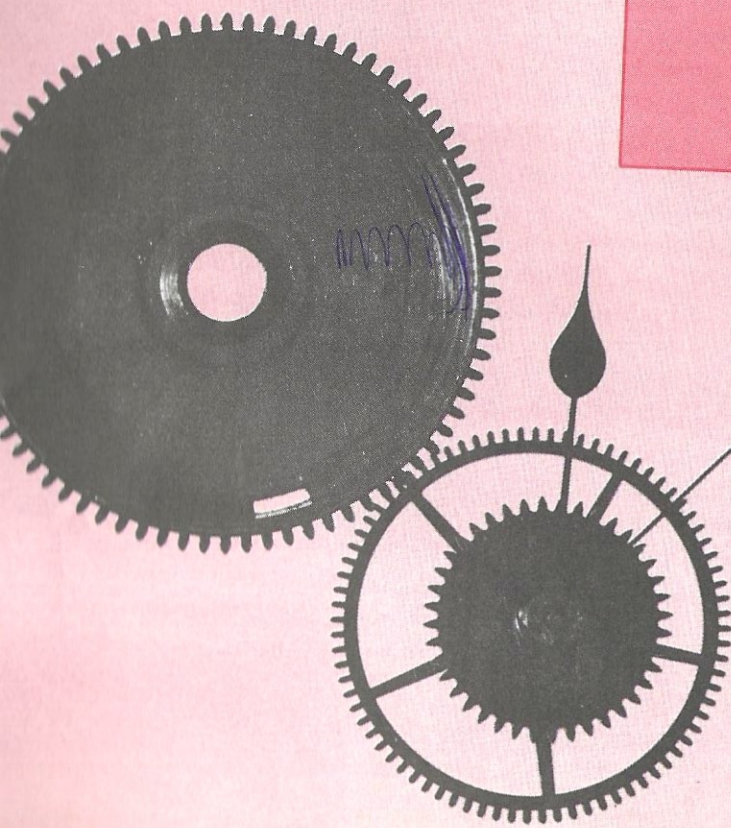
1. Solve $6x^2 + 13x + 6 = 0$ by following the flow chart.
2. Solve $x^2 - 2x + 2 = 0$ by following the flow chart.
3. Suppose you want to solve an equation of the form $ax^2 + bx = 0$, $a \neq 0$, $b \neq 0$. You could solve it by using the flow chart for the general quadratic equation, but it is really too easy for that. Write a flow chart for solving $ax^2 + bx = 0$. (Remember that the computer will not factor $ax^2 + bx = 0$ into $x(ax + b) = 0$. You must only ask the machine to do the standard operations.)
4. You know that if the value of $b^2 - 4ac$, the discriminant, is less than zero, there are no real solutions to the quadratic equation. It is a waste of computer time to go through the entire program if there are no real-number solutions. Insert a step into the flow chart that will eliminate some steps and eliminate the time wasted.

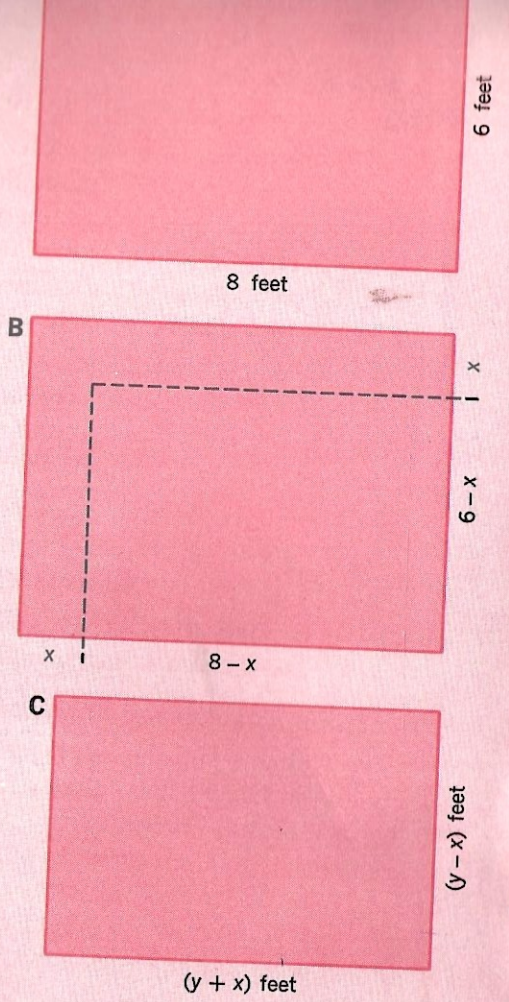
THE GROWTH OF MATHEMATICS

When Mathematics Almost Disappeared

The Romans and the Greeks were very different in temperament. Although the Greeks had cultivated mathematics for its own sake, the Romans were only interested in "practical" mathematics. They paid little attention to the correct results of the Greeks, and as a consequence, many of the formulas they used were wrong. As the Roman Empire spread over most of the civilized world, so did the Roman attitude toward mathematics. With the gradual decline of the Empire, partial ignorance of mathematics gave way to even greater ignorance.

In the Middle Ages there were a few attempts to revive some of the Greek spirit in mathematics. Boethius (A.D. 480?–?524), born just after the fall of the Roman Empire, wrote an arithmetic and a geometry. Part of the geometry is a restatement of a few theorems from Euclid (without proof) and the remainder consists of the mostly incorrect formulas used by the Romans. For the next five or six centuries, these were about the only books on mathematics known in Europe. During this time most computation was done by means of finger-counting or using one form or another of the abacus. (The abacus could be used without any knowledge of mathematics.) In the monasteries there was sufficient arithmetic taught to enable the monks to calculate the dates of religious feasts. This state of affairs persisted until the twelfth century, when Arabic mathematics began to be known in Europe.





CHAPTER 9
RATIONAL EXPRESSIONS

● Diagram *A* at the left represents the floor of the elevator in the photograph. The dimensions of this rectangular floor are 8 feet by 6 feet. By counting, you know that there are seven people in the elevator. How would you find the number of square feet of floor space for each person in the elevator? Does $\frac{8 \times 6}{7}$ show the number of square feet of floor for each person? Name $\frac{8 \times 6}{7}$ as the ratio of two integers.

● Suppose the same seven people were to stand in an elevator with a smaller floor, as represented in *B*. What are the length and width of the smaller floor? Why does $\frac{(8 - x)(6 - x)}{7}$ show the number of square feet of floor space for each person? Rename $\frac{(8 - x)(6 - x)}{7}$ so that the numerator is a polynomial.

● What expression would you use if you want the number of square feet of floor space per person for any number of people, y, standing on floor *B*? for any number of people, y, standing on floor *C*?

● The radius of the larger gear in the photograph is a polynomial, $\frac{1}{4}x$. The radius of the smaller gear is the quotient of two polynomials, $\frac{x + 5}{15 - x}$. What is the ratio of the radius of the larger gear to the radius of the smaller?

● Write an expression for the area of the larger gear; for the area of the smaller gear; for the sum of the areas of both gears.

Each of the expressions used above to name the number of square feet per person and the expressions used with respect to the gears are rational expressions. In this chapter you will learn the meaning of rational expressions and how to perform operations on them.

9–1 Definition and Equality of Rational Expressions

Look at the examples below and try to formulate the definition of **rational expression.**

Rational expression	Nonrational expression
a. $\frac{480}{\sqrt{13}}$	b. $\frac{\sqrt[3]{13y}}{1}$
c. $\frac{x^2 - \sqrt{3}x + 480}{480}$	d. $\frac{x^2 - \sqrt{5x} + 8}{480}$
e. $\frac{y^2 - x^2}{y - x}$	f. $\frac{y + \sqrt{x}}{y + \sqrt{x}}$
g. $\frac{4x^3 - 3x^2 - 2x + 1}{x^2 + 3x - 9}$	h. $\frac{3x^{\frac{1}{2}} - 9}{x + 3}$

Definition A rational expression is one that can be expressed as the ratio of two polynomials, $\frac{P}{Q}$, where the numerical coefficients of Q are not all zero.

Tell why each expression in the table above either fits or does not fit the definition of rational expression. How can the definition of rational number help you remember the definition of rational expression? Tell how they are alike and how they are different.

If two rational numbers have either the same numerator or the same denominator, it is easy to tell whether or not the two rational numbers are equal.

$\frac{2}{3} \neq \frac{1}{3}$ because $2 \neq 1$
$\frac{5}{6} \neq \frac{5}{7}$ because $6 \neq 7$

In a similar way, equality of rational expressions with the same numerator or denominator can be determined. For example, for all real numbers for x,

$$\frac{x^2 - 5}{6} \neq \frac{x^2 - 4}{6} \quad \text{because} \quad x^2 - 5 \neq x^2 - 4.$$

For rational numbers with different numerators and denominators, equality is defined in a different way.

$\frac{3}{4} = \frac{6}{8}$ because $3 \times 8 = 4 \times 6$

Equality of any two rational expressions is defined in the same way.

Definition Two rational expressions $\frac{Q}{R}$ and $\frac{S}{T}$ are equal if and only if $Q \cdot T = R \cdot S$.

EXAMPLE. Use the definition of equality of rational expressions to show that $\frac{x^2 - 4}{x - 2}$ and $\frac{x + 2}{1}$ are equal.

$$\frac{x^2 - 4}{x - 2} \stackrel{?}{=} \frac{x + 2}{1}$$
$$(x^2 - 4)1 \stackrel{?}{=} (x - 2)(x + 2)$$
$$x^2 - 4 = x^2 - 4$$

Thus, $\frac{x^2 - 4}{x - 2}$ and $\frac{x + 2}{1}$ are equal. If x is 2, then $\frac{x^2 - 4}{x - 2}$ is not defined. Equality for rational expressions, however, was defined in such a way that the expressions are technically equal even when one (or both) are undefined for particular values of the variables.

Now try these

Use the definition of equality, and replace ● with $=$ or $\neq$ to make true sentences.

1. $\frac{4m}{6m^2}$ ● $\frac{2}{3m}$
2. $\frac{x}{y}$ ● $\frac{xa}{ya}$
3. $\frac{x - y}{x}$ ● $\frac{-y}{1}$
4. $\frac{x - y}{2}$ ● $\frac{(x - y)(x + y)}{2(x + y)}$
5. $\frac{5x^2}{3x}$ ● $\frac{5x}{3}$
6. $\frac{a^2 + b^2}{a + b}$ ● $\frac{a + b}{1}$

Answers: **1.** $=$ **2.** $=$ **3.** $\neq$ **4.** $=$ **5.** $=$ **6.** $\neq$

Exercises

A Write *Rational* or *Nonrational* to describe each of the following expressions.

1. $\frac{x^2 + 3x}{2}$
2. $\frac{y^3}{x^2}$
3. $\frac{\sqrt{4x}}{5}$
4. $\frac{(xy)^{\frac{1}{2}}}{x}$
5. $\sqrt{\frac{s}{t}}$
6. $(x^2 - 9)^{\frac{1}{4}}$
7. $\frac{(x^3 - 1)^2}{x + y}$
8. $\frac{r^2 - r - 3}{\sqrt{y}}$
9. $\frac{(y - 2)(y + 3)}{y^2}$

Using the definition of equality of rational expressions, replace ● with = or ≠ to make true sentences.

10. $\frac{5}{8} \bullet \frac{15}{24}$
11. $\frac{3}{7} \bullet \frac{6}{13}$
12. $\frac{\sqrt{12}}{5} \bullet \frac{\sqrt{3}}{2\frac{1}{2}}$
13. $\frac{15}{22} \bullet \frac{7 \cdot 5}{11}$
14. $\frac{2\sqrt{2}}{9} \bullet \frac{\sqrt{2}}{4}$
15. $\frac{\sqrt{6}}{\sqrt{15}} \bullet \frac{\sqrt{2}}{\sqrt{5}}$
16. $\frac{x^2}{2} \bullet \frac{x^3}{2x}$
17. $\frac{x+2}{y} \bullet \frac{x+2}{xy}$
18. $\frac{r^2-6r+9}{r-3} \bullet \frac{r+3}{1}$
19. $\frac{2a^2}{3a} \bullet \frac{4a^3}{6a}$
20. $\frac{5b^3c}{2b} \bullet \frac{2 \cdot 5b^2c^2}{c}$
21. $\frac{-4xy}{16x^2y^3} \bullet \frac{1}{4xy^2}$
22. $\frac{x^2-4}{2x-4} \bullet \frac{x+2}{2}$
23. $\frac{y^2-y-2}{2y^2-5y+2} \bullet \frac{y+1}{2y-1}$

Add 7 to the numerator and denominator of each rational expression. Write a true sentence using = or ≠ to compare the original expression with the new one.

24. $\frac{5}{6}$
25. $\frac{x^2}{x}$
26. $\frac{x+4}{x}$
27. $\frac{x^2-7}{x-7}$

Multiply the numerator and denominator of each rational expression by $x - 2$. Write a true sentence using = or ≠ to compare the original expression with the new one.

28. $\frac{x^3}{x}$
29. $\frac{x+2}{x-2}$
30. $\frac{x^2-4}{x+4}$
31. $\frac{x^2}{y^2}$

9–2 Simplifying Rational Expressions

Using the definition of equality, it is easy to prove that a theorem you have used for rational numbers also applies to rational expressions.

Theorem 9–1 The numerator and denominator of a rational expression may be multiplied or divided by the same nonzero polynomial without changing the value of the expression.

$$\frac{P}{Q} = \frac{RP}{RQ}$$

By the definition of equality, the two rational expressions are equal if $P(RQ) = Q(RP)$ is true.

Statements	*Reasons*
1. $P(RQ) = (PR)Q$	1. Associative postulate of multiplication.
2. $(PR)Q = Q(PR)$	2. Commutative postulate of multiplication.
3. $Q(PR) = Q(RP)$	3. Commutative postulate of multiplication.

Thus, $P(RQ) = Q(RP)$ is true and $\frac{P}{Q} = \frac{RP}{RQ}$ is true.

EXAMPLE 1. Express $\frac{1}{2}$ as a rational number whose denominator is 6.

METHOD I

Use the multiplicative property of 1 to make the denominator 6.

$$\frac{1}{2} = 1 \times \frac{1}{2} = \frac{3}{3} \times \frac{1}{2} = \frac{3 \times 1}{3 \times 2} = \frac{3}{6}$$

METHOD II

Think: By what number is 2 multiplied to get 6? Then, multiply 1 by the same number, 3.

$$\frac{1}{2} = \frac{3 \times 1}{3 \times 2} = \frac{3}{6}$$

Check $1 \times 6 = 2 \times 3$
$6 = 6$

EXAMPLE 2. Express $\frac{4m}{6m^2}$ so that the numerator and denominator have only 1 as a common factor.

The process can be thought of in two ways.

METHOD I

Use the property of 1.

$$\frac{4m}{6m^2} = \frac{2m \cdot 2}{2m \cdot 3m} = \frac{2m}{2m} \cdot \frac{2}{3m} = 1 \cdot \frac{2}{3m} = \frac{2}{3m}$$

METHOD II

Factor the numerator and denominator and then divide both by the same number.

$$\frac{4m}{6m^2} = \frac{2m \cdot 2}{2m \cdot 3m} = \frac{2}{3m}$$

Check $4m \cdot 3m \stackrel{?}{=} 6m^2 \cdot 2$
$12m^2 = 12m^2$

The expression $\frac{2}{3m}$ is the simplest form of the rational expression $\frac{4m}{6m^2}$. A rational expression is in its **simplest form** when both polynomials of the rational expression have no common factors other than 1 or -1. To **simplify** an expression means to write the expression in its simplest form.

In the following example, the numerator and denominator are first written in factored form so the common factors are easily seen.

EXAMPLE 3. Simplify $\frac{x^2+xy}{x^2-xy}$.

Factor the numerator and denominator completely, and then divide both by the common factor.

$$\frac{x^2+xy}{x^2-xy} = \frac{x(x+y)}{x(x-y)} = \frac{x+y}{x-y}$$

There are no common factors of $x+y$ and $x-y$; so $\frac{x+y}{x-y}$ is the simplest form of $\frac{x^2+xy}{x^2-xy}$. Check to see that they are equal.

Remember that you always divide both numerator and denominator by the same *factor*. The rational expression $\frac{2a+3}{a}$ cannot be simplified because $2a+3$ and a have no common factor; that is, a is not a factor of the sum $2a+3$.

Now try these

— Is the second expression a factor of both the numerator and denominator of the rational expression? When your answer is *Yes*, simplify the rational expression and check, using the definition of equality.

1. $\frac{3\cdot 5}{3\cdot 6}$; 3
2. $\frac{a\cdot x}{a\cdot y}$; a
3. $\frac{3+7}{3}$; 3
4. $\frac{a+5}{a}$; a
5. $\frac{x(x-1)}{x}$; x
6. $\frac{3y^2-y}{y}$; y
7. $\frac{6x-6y}{6x+6y}$; x
8. $\frac{6x-6y}{6x+6y}$; 6
9. $\frac{x^3-3}{x^3}$; x^3

Answers: **1.** Yes; $\frac{3\cdot 5}{3\cdot 6}=\frac{5}{6}$; $(3\cdot 5)\cdot 6=(3\cdot 6)\cdot 5$ **2.** Yes; $\frac{a\cdot x}{a\cdot y}=\frac{x}{y}$; $(ax)y=(ay)x$ **3.** No. **4.** No. **5.** Yes; $\frac{x(x-1)}{x}=\frac{(x-1)}{1}$; $x(x-1)\cdot 1=x(x-1)$ **6.** Yes; $\frac{3y^2-y}{y}=\frac{y(3y-1)}{y}=\frac{3y-1}{1}$; $(3y^2-y)\cdot 1=y(3y-1)$ **7.** No. **8.** Yes; $\frac{6(x-y)}{6(x+y)}=\frac{x-y}{x+y}$; $[6(x-y)](x+y)=[6(x+y)](x-y)$ **9.** No.

Exercises

A — Replace the ? by an expression to make a true sentence.

1. $\frac{8}{12}=\frac{?}{60}$
2. $\frac{24}{44}=\frac{6}{?}$
3. $\frac{6ab}{18a^2}=\frac{b}{?}$
4. $\frac{2x}{?}=\frac{x}{y}$
5. $\frac{?}{6a}=\frac{4}{3a}$
6. $\frac{2a}{a+b}=\frac{?}{6(a+b)}$
7. $\frac{b}{xy}=\frac{?}{ax^2y^2}$
8. $\frac{c-2}{2c+5}=\frac{c^2-2c}{?}$
9. $\frac{(x+y)(x-y)}{(x+y)(x+y)}=\frac{?}{x+y}$

— Simplify each of the following expressions.

Example. $\frac{6n^2}{15nm}$

Divide numerator and denominator by the common factor.

$$\frac{6n^2}{15nm}=\frac{3n\cdot 2n}{3n\cdot 5m}=\frac{2n}{5m}$$

10. $\frac{3\cdot 5}{3\cdot 4}$
11. $\frac{3\cdot 4\cdot 5}{3\cdot 4}$
12. $\frac{6xy}{6}$
13. $\frac{6xy}{6x}$
14. $\frac{12}{18}$
15. $\frac{9}{12}$
16. $\frac{a^2}{ab}$
17. $\frac{24a}{36b}$
18. $\frac{12x}{15xy}$
19. $\frac{25x^2}{30x}$
20. $\frac{18x^2y}{24xy}$
21. $\frac{3xy}{24y}$
22. $\frac{y^7}{y^3}$
23. $\frac{y^5}{y^3}$
24. $\frac{30s^2}{20s}$
25. $\frac{7x^2}{6x}$
26. $\frac{r^2xy}{r^2xt}$
27. $\frac{6ab}{9a^2x}$
28. $\frac{2a^3b^2}{6a^2b^2}$
29. $\frac{-x^2y^3}{x^3y^2}$
30. $\frac{3a^4b^3}{9a^3b^3}$
31. $\frac{48ab^2c}{60a^2b^2}$
32. $\frac{-9xy}{30x^2y}$
33. $\frac{3\cdot 5x}{2\cdot 5y}$
34. $\frac{-30a^2bc}{-18a^3c}$
35. $\frac{36a^2b^3x^2}{-60a^2bx^5}$
36. $\frac{a(x+y)}{b(x+y)}$
37. $\frac{24}{15(x+y)}$
38. $\frac{m(x-y)^2}{e(x-y)}$
39. $\frac{30a}{25(x-a)}$
40. $\frac{36x}{15(x+2)}$
41. $\frac{15(a+b)}{6(2a+b)}$
42. $\frac{8(x-2y)}{18(2x-y)}$

Example. $\frac{3x^2-xy}{xy+3x^2}$

$$\frac{3x^2-xy}{xy+3x^2}=\frac{x(3x-y)}{x(y+3x)}=\frac{3x-y}{y+3x}$$

43. $\frac{3x-6y}{9x+12y}$
44. $\frac{a^2b-ab^2}{a^2b^2+ab^2}$
45. $\frac{2ab+b^2}{a^2-b^2}$

46. $\frac{6x - 6y}{6a + 6b}$ 47. $\frac{ax + by}{ax - by}$ 48. $\frac{2x - 10y}{2a - 2b}$

Tell whether the following statements are *True* or *False*.

49. $\forall x \neq 0 \quad \frac{6x^2}{x} = 6x$

50. $\forall x \neq 0 \quad \frac{8x^3}{2x^2} = 4x$

51. $\forall y \neq 2 \quad \frac{y}{y^2 - 2y} = \frac{1}{y - 2}$

52. $\forall a \neq \pm 3 \quad \frac{2a + 6}{a^2 - 9} = \frac{2}{a + 3}$

53. $\forall x \neq \pm\sqrt{6} \quad \frac{x^3 - 6x}{x^2 - 6} = x$

54. $\forall x, y \neq 0 \quad \frac{3x^4y^2}{10x^2y^2} = \frac{3x^2}{10y}$

B Simplify the following expressions.

Example. $\frac{a^2 - 9}{a^2 + 5a + 6}$

First factor, and then divide the numerator and denominator by the common factor.

$$\frac{a^2 - 9}{a^2 + 5a + 6} = \frac{(a - 3)(a + 3)}{(a + 2)(a + 3)} = \frac{a - 3}{a + 2}$$

55. $\frac{x^2 - y^2}{x^2 - 2xy + y^2}$

56. $\frac{3a + 3b}{a^2 + 2ab + b^2}$

57. $\frac{2x^2 + 6x + 4}{4x^2 - 12x - 16}$

58. $\frac{2x^3 - 2x^2 - 4x}{3x^3 + 3x^2 + 18x}$

59. $\frac{x^2 - 3x - 4}{2x^2 + 10x + 8}$

60. $\frac{2x^2 - 2x - 12}{x^3 - 5x^2 + 6x}$

61. $\frac{x^2 - 25}{x^2 - 10x + 25}$

62. $\frac{5a + 5}{a^2 + 7a + 6}$

63. $\frac{6x + 12}{x^2 - x - 6}$

64. $\frac{x^2y - y}{x + 1}$

65. $\frac{3a^2 - 3}{a - 1}$

66. $\frac{a^2 + 1}{a + 1}$

C Express the following in simplest form.

67. $\frac{R^2 - r^2}{3r + 3R}$

68. $\frac{6a^2 + a - 15}{6a^2 - 13a + 6}$

69. $\frac{3a^2 - 27}{6a^2 + 15a - 9}$

70. $\frac{6a^2 + 42a + 72}{18 - 2a^2}$

71. $\frac{3a^2 - 12a - 15}{18 - 18a^2}$

72. $\frac{3x^2 - 27}{24 - 11x + x^2}$

73. $\frac{x^2 + 6xy + 9y^2}{x^2 - 9y^2}$

74. $\frac{16a^2 - 48ab + 36b^2}{ba^2 - 36b^2}$

75. $\frac{a^2 - 4a + 4}{4 - a^2}$

76. $\frac{4x^n(a + b)}{8x^{n-1}(a + b)^n}$

9–3 Multiplication of Rational Expressions

To multiply two rational numbers, you find the product of the numerators to determine the numerator of the product and you find the product of the denominators to determine the denominator of the product. The same holds true for rational expressions. The first theorem treats rational expressions with 1 in the numerator.

Theorem 9–2 The product of $\frac{1}{Q}$ and $\frac{1}{S}$, where $\frac{1}{Q}$ and $\frac{1}{S}$ are rational expressions, is $\frac{1}{QS}$.

The proof is left as an exercise.

Then the general theorem for multiplication of rational expressions can be stated.

Theorem 9–3 The product of $\frac{P}{Q}$ and $\frac{R}{S}$ is $\frac{PR}{QS}$.

$$\forall Q \neq 0 \quad \forall S \neq 0 \quad \frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$$

The proof is left as an exercise. Is the product of two rational expressions always a rational expression? Explain.

EXAMPLE 1. Find the product of $\frac{2x}{y}$ and $\frac{w}{3z}$.

$$\frac{2x}{y} \cdot \frac{w}{3z} = \frac{2xw}{3yz}$$

Remember to express the product in simplest form.

EXAMPLE 2. Simplify the product $\frac{3a^2b}{2ab^2} \cdot \frac{4abx}{9a^2by}$.

You can multiply the terms and then simplify the product or simplify and then multiply.

METHOD I

$$\frac{3a^2b}{2ab^2} \cdot \frac{4abx}{9a^2by} = \frac{12a^3b^2x}{18a^3b^3y} = \frac{(6a^3b^2)2x}{(6a^3b^2)3by} = \frac{2x}{3by}$$

METHOD II

$$\frac{3a^2b}{2ab^2} \cdot \frac{4abx}{9a^2by} = \frac{1}{\cancel{3a^2b}} \cdot \frac{2x}{\cancel{4abx}} = \frac{2x}{3by}$$

Can you see why only Method II from the previous page is used in Example 3 below?

EXAMPLE 3. Simplify $\frac{3a+3b}{a-b} \cdot \frac{a+2b}{a^2+2ab+b^2}$.

$$\frac{3a+3b}{a-b} \cdot \frac{a+2b}{a^2+2ab+b^2} = \frac{3(a+b)(a+2b)}{(a-b)(a+b)(a+b)}$$

$$= \frac{3(a+2b)}{(a-b)(a+b)}, \text{ or } \frac{3a+6b}{a^2-b^2}$$

You can see that the numerator and the denominator of each fraction were factored completely to make it easier to name the indicated product in lowest terms. When a product contains polynomial factors, as in Example 3, the answer may be left in either form shown above.

Now try these

Find the product in simplest form.

1. $\frac{a^2}{b^2c} \cdot \frac{ab^2}{a}$
2. $\frac{5ab}{3x} \cdot \frac{12x^2}{10a^2b}$
3. $\frac{a+b}{a} \cdot \frac{a^2}{b}$
4. $\frac{m^2-n^2}{m+n} \cdot \frac{m+n}{m-n}$
5. $\frac{x^2-xy}{x-y} \cdot \frac{1}{x}$
6. $\frac{x-2}{x-5x+6} \cdot \frac{x^2-9}{x^2-3x}$

Answers: 1. $\frac{a^2}{c}$ 2. $\frac{2x}{a}$ 3. $\frac{a(a+b)}{b}$ 4. $m+n$ 5. 1 6. $\frac{x+3}{x^2-3x}$

Exercises

A Write the product in simplest form.

1. $\frac{x}{y} \cdot \frac{my}{nx}$
2. $\frac{ab}{xy} \cdot \frac{x}{a^3b^2}$
3. $\frac{xy^2}{m^3n} \cdot \frac{m^2}{y^3}$
4. $\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{ax}{by} \cdot \frac{y^2}{x^2}$
5. $\frac{7xy^2}{8x^2y} \cdot \frac{16xz^2}{49y^2z}$
6. $\frac{3}{x} \cdot \frac{x^2}{3}$
7. $\frac{4 \cdot 6}{2 \cdot 3^2} \cdot \frac{27}{2^2 \cdot 3}$
8. $\frac{3x}{d} \cdot \frac{c}{3x}$
9. $\frac{ab^2}{2} \cdot \frac{c}{abd}$
10. $(x+2) \cdot \frac{x+3}{x+1}$
11. $(2x+5) \cdot \frac{2x+3}{2x-5}$
12. $\frac{(x-5)^2}{3(x-5)}$
13. $x^2y \cdot \frac{8}{2x+2xy}$
14. $\frac{3a-3b}{10ab} \cdot \frac{50a^2b^2}{a^2-b^2}$
15. $\frac{a+1}{a+2} \cdot \frac{a+2}{a+1}$
16. $(n+3) \cdot \frac{n+3}{n+5}$
17. $(2x-5) \cdot \frac{x+7}{2x-5}$
18. $\frac{a+3}{a-3} \cdot \frac{1}{a+3}$
19. $n(a+b) \cdot \frac{1}{m(a+b)}$
20. $\frac{a^2-4b^2}{a^2-b^2} \cdot \frac{3a^2b^2}{a+b}$
21. $\frac{a^2+5a}{a^2-16} \cdot \frac{a^2-4a}{a^2-25}$
22. $\frac{3x-48y^2}{2x^2-8y^2} \cdot \frac{3x+6y}{3x+12y}$
23. $\frac{x^2+8x+16}{x^2-9} \cdot \frac{x-3}{x+4}$
24. $\frac{a^2-3a-10}{(a-2)^2} \cdot \frac{a-2}{a-5}$

25. $\frac{9-x^2}{x+3} \cdot \frac{x}{3-x}$
26. $\frac{24x^3}{3(x^2-4x+4)} \cdot \frac{3x-6}{2x}$
27. $\frac{x^2-6x+5}{x-1} \cdot \frac{x-1}{x-5}$
28. $\frac{c^2-6c}{c-6} \cdot \frac{c+3}{c}$
29. $\frac{x^2-24-2x}{x^2-30-x} \cdot \frac{x+5}{x^2-16}$
30. $\frac{9-y^2}{r^2-r} \cdot \frac{r-1}{y+3}$

B Write the product in simplest form.

31. $\frac{a^2+7ab+10b^2}{a^2+6ab+5b^2} \cdot \frac{a+b}{a^2+4ab+4b^2} \cdot \frac{a+2b}{1}$
32. $\frac{x^2-y^2}{x^2-3xy+2y^2} \cdot \frac{xy-2y}{y^2+xy} \cdot \frac{x(x-y)}{(x-y)^2}$
33. $\frac{2a-3b}{a^2+4ab+4b^2} \cdot \frac{4a^2-4b^2}{4a^2-9b^2} \cdot \frac{5a^2+10ab}{3ab-3b^2}$
34. $\frac{6x-3y}{4x^2+4xy+y^2} \cdot \frac{2x+y}{4x^2-4xy+y^2}$
35. $\frac{5m+5n}{m^2-n^2} \cdot \frac{m^2-mn}{(m+n)^2}$
36. $\frac{x^2-3x-18}{x^2-x-2} \cdot \frac{3x+3}{x^2-2x-15}$
37. $\frac{a^2+6a+5}{a-3} \cdot \frac{5a-15}{a^2+4a-5}$
38. $\frac{3a+9}{3a^2-5a+2} \cdot \frac{6a^2-7a+2}{2a^2+3a-18}$
39. $\frac{4x^2-9y^2}{6x^2-9xy} \cdot \frac{6xy}{4xy+6y^2}$
40. $\frac{x^2+6x+8}{x^2+4x+14} \cdot \frac{x^2+5x+6}{x^2+7x+12}$
41. $\frac{a^2+a-2}{a^2-3a-10} \cdot \frac{a^2-2a-8}{a^2-7a+12}$
42. $\frac{x^2-2xb+b^2}{b-1} \cdot \frac{b^2-5b+4}{x^2-b^2}$

C Write the product in simplest form.

43. $\frac{x^2-x-1}{x-1} \cdot \frac{x^2-1}{x} \cdot \frac{x^2-x-1}{x-1}$
44. $\frac{a^2-5a+6}{6a^2-17a+5} \cdot \frac{6a^2+7a-3}{2a^2-7a+3} \cdot \frac{2a^2-7a+5}{a^2-3a+2}$
45. Begin the proof of Theorem 9–2 by multiplying each side of the equation by its multiplicative inverse QS. If both products equal 1, the multiplicative identity element, then $\frac{1}{Q} \cdot \frac{1}{S}$ and $\frac{1}{QS}$ have the same multiplicative inverse and they must be equal. Supply the reasons in the following proof.

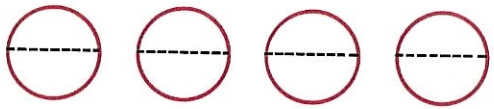
Statements	*Reasons*
1. $(QS)\left(\frac{1}{Q}\right)\left(\frac{1}{S}\right) = \left[Q\left(\frac{1}{Q}\right)\right] \times \left[S\left(\frac{1}{S}\right)\right]$	1. Associative and commutative postulates of multiplication.
2. $= 1 \times 1$, or 1	2. ?
3. $(QS)\left[\frac{1}{QS}\right] = 1$, $QS \neq 0$	3. ?
4. Thus, $\frac{1}{Q} \cdot \frac{1}{S}$ and $\frac{1}{QS}$ are equal.	4. ?

46. Theorem 9–3 can be proved by beginning with the left side of the equation and working to make it equal to the right side. Fill in the missing reasons.

Statements	*Reasons*
1. $\frac{P}{Q}\cdot\frac{R}{S}=\left(P\cdot\frac{1}{Q}\right)\left(R\cdot\frac{1}{S}\right)$	1. Postulate of division.
2. $=P\cdot\left(\frac{1}{Q}\cdot R\right)\cdot\frac{1}{S}$	2. Associative postulate (applied twice).
3. $=P\cdot\left(R\cdot\frac{1}{Q}\right)\cdot\frac{1}{S}$	3. ?
4. $=PR\left(\frac{1}{Q}\cdot\frac{1}{S}\right)$	4. ?
5. $=PR\left(\frac{1}{QS}\right)$	5. ?
6. $\frac{P}{Q}\cdot\frac{R}{S}=\frac{PR}{QS}$	6. ?

9–4 Division of Rational Expressions

It is easy to draw a diagram to show that $4 \div \frac{1}{2} = 8$ is true, or that there are eight halves in 4.



To make a diagram to illustrate the quotient for $\frac{2}{3} \div \frac{5}{7}$ would be more difficult. But the computation can be done easily by using the properties you have learned.

$$\frac{2}{3} \div \frac{5}{7} = \frac{\frac{2}{3}}{\frac{5}{7}} = \frac{\frac{2}{3}\cdot\frac{7}{5}}{\frac{5}{7}\cdot\frac{7}{5}}$$

Simplifying makes the denominator 1.

$$\frac{\frac{2}{3}\cdot\frac{7}{5}}{1} = \frac{2}{3}\cdot\frac{7}{5} = \frac{14}{15}$$

Then,

$$\underbrace{\frac{2}{3} \div \frac{5}{7}}_{\text{division by } \frac{5}{7}} = \underbrace{\frac{14}{15}}_{\text{is the same as}} = \underbrace{\frac{2}{3}\cdot\frac{7}{5}}_{\text{multiplication by } \frac{7}{5}}$$

Notice that you multiplied by $\frac{7}{5}$, the reciprocal of the divisor $\frac{5}{7}$. In this way, the division process becomes a multiplication process. The same idea holds for division of rational expressions.

Theorem 9–4 To divide a rational expression $\frac{P}{Q}$ by a rational expression $\frac{R}{S}$, multiply $\frac{P}{Q}$ by the multiplicative inverse, or reciprocal, of $\frac{R}{S}$.

The proof will be left as an exercise.

Now try these

Write the quotient in simplest form.

1. $\frac{a}{b} \div \frac{a}{c}$ **2.** $\frac{x^2y}{x} \div \frac{x}{y^2}$ **3.** $\frac{5a^2}{7b} \div \frac{15a^3}{14b^2}$ **4.** $\frac{x-y}{x+y} \div \frac{x^2-y^2}{2}$

Answers: **1.** $\frac{c}{b}$ **2.** y^3 **3.** $\frac{2b}{3a}$ **4.** $\frac{2}{x^2+2xy+y^2}$

Exercises

A Find the quotients. Express answers in lowest terms.

1. $\frac{x}{2} \div \frac{x}{4}$ **2.** $\frac{x}{4} \div \frac{x}{a}$ **3.** $\frac{2}{x} \div \frac{4}{x}$

4. $\frac{4}{x} \div \frac{2}{x}$ **5.** $\frac{x}{y} \div \frac{3}{y}$ **6.** $\frac{ad}{bc} \div \frac{a}{b}$

7. $\frac{a}{b} \div \frac{c}{d}$ **8.** $\frac{xy}{ab} \div \frac{xy}{bc}$ **9.** $\frac{1}{a} \div \frac{a}{b}$

10. $\frac{a}{x} \div \frac{c}{x}$ **11.** $\frac{1}{f} \div \frac{1}{g}$ **12.** $\frac{a}{x} \div \frac{x}{c}$

13. $\frac{8}{3a} \div 16$ **14.** $\frac{5x^2}{8xy} \div 15x$ **15.** $\frac{9ab^2}{8xy^2} \div \frac{3b^2}{2xy}$

16. $\frac{14a^2}{10b^2} \div \frac{21a^2}{15b^2}$ **17.** $\frac{5x}{12yz^2} \div \frac{15x^3}{18y^2z^2}$ **18.** $\frac{25x^2}{(3yz)^2} \div \frac{100x^3}{18yz^3}$

B Find the quotients. Express answers in lowest terms.

19. $\frac{x^2+x}{a} \div x^2$ **20.** $\frac{(a-b)(a+b)}{(x-y)(x+y)} \div \frac{a-b}{x+y}$

21. $\frac{a^2-b^2}{x^2-y^2} \div \frac{a+b}{x-y}$ **22.** $\frac{(2x)^3}{(4yz)^3} \div \frac{16x^2}{8y^2z^3}$

23. $\dfrac{3(x+y)^2}{x-y} \div 6(x+y)$

24. $\dfrac{x-y}{x+y} \div \dfrac{5x^2-5y^2}{3x-3y}$

25. $\dfrac{x^2+2x+1}{3x} \div (x+1)$

26. $\dfrac{a^3-6a^2+8a}{5x} \div \dfrac{2a-4}{10a-40}$

27. $\dfrac{5a^2-5ab}{ab+b^2} \div \dfrac{5a^2-5b^2}{b}$

28. $\dfrac{4x^2-4xy-3y^2}{3x^2y} \div \dfrac{2x^2-3xy}{6x^3}$

C — Perform the indicated operations.

29. $\dfrac{x^2-y^2}{x^2} \cdot \dfrac{x+y}{x-y} \div \dfrac{x^2-y^2}{yx}$

30. $\dfrac{a^2-2ab+b^2}{ab} \cdot \dfrac{a+b}{a-b} \div \dfrac{a^2-b^2}{a^2}$

31. $\dfrac{a^3b^3}{a^3-ab^2} \div \dfrac{abc}{a-b} \cdot \dfrac{ab+bc}{ab}$

32. $\dfrac{xy-xz-x^2}{xyz} \cdot \dfrac{x^2y^2z^3}{yx-zx} \div x^3$

33. $\dfrac{2x^3}{2a+b} \div \dfrac{10a^2}{4a^2+4ab+b^2} \cdot \dfrac{12a+3b}{2a^2+ab}$

34. Prove Theorem 9–4. (*Hint:* Follow the numerical example on page 396.)

9–5 Least Common Multiple of Polynomials

Consider the rational numbers 4 and 6. The natural-number multiples of 4 are

$$\{4, 8, 12, 16, 20, 24, 28, 32, 36, \cdots\}.$$

The natural-number multiples of 6 are

$$\{6, 12, 18, 24, 30, 36, \cdots\}.$$

The multiples common to both 4 and 6 are

$$\{12, 24, 36, \cdots\}.$$

What is the least multiple in this last set? The **least common multiple** (LCM) of 4 and 6 is 12, because 12 is the least multiple of both 4 and 6; 12 is the least number that has both 4 and 6 as factors.

The same idea is used to define the least common multiple of polynomials.

Definition The least common multiple (LCM) of two or more polynomials is the least polynomial that has each of the given polynomials as factors.

To find the LCM easily, first write each polynomial in factored form. Decide which factors are common to both polynomials. The factors that are not common factors are the "remaining" factors. The LCM is the product of all the common factors and the remaining factors.

EXAMPLE 1. Find the LCM of 8 and 12.

$$8 = 2 \cdot 2 \cdot 2$$
$$12 = 2 \cdot 2 \cdot 3$$

The factors common to both numbers are 2 and 2. The remaining factors are 2 and 3. The LCM, the product of these factors, is $2 \cdot 2 \cdot 2 \cdot 3$, or 24. The least number that has both sets of factors in it is 24.

EXAMPLE 2. Find the LCM of $6x^4y$ and $9x^2y^2$.

$$6x^4y = 2 \cdot 3 \cdot x \cdot x \cdot x \cdot x \cdot y$$
$$9x^2y^2 = 3 \cdot 3 \cdot x \cdot x \cdot y \cdot y$$

The common factors are $3 \cdot x \cdot x \cdot y$, or $3x^2y$.
The other factors are $2 \cdot x \cdot x \cdot 3 \cdot y$, or $6x^2y$.
Thus, the LCM is $3x^2y \cdot 6x^2y$, or $18x^4y^2$.

EXAMPLE 3. Find the LCM of $x^2 - 5x + 6$ and $x^2 + 2x - 8$.

$$x^2 - 5x + 6 = (x-3)(x-2)$$
$$x^2 + 2x - 8 = (x+4)(x-2)$$

The common factor is $(x-2)$. The remaining factors are $(x-3)$ and $(x+4)$. The LCM is $(x-2)(x-3)(x+4)$, or $x^3 - x^2 - 14x + 24$.

Now try these

— Find the least common multiple for each of the following.

1. $6xy, 9x^2$

2. $14, 2s^3t, 4t^3$

3. $8a^2b, 6ab^3, 12abc$

4. $(x^3-4), (x^3+3x+2)$

Answers: **1.** $18x^2y$ **2.** $28s^3t^3$ **3.** $24a^3b^3c$ **4.** $(x-2)(x+2)(x+1)$, or x^3+x^2-4x-4

Exercises

A — Find the LCM of the following.

1. 6 and 2

2. 3 and 4

3. 2 and 5

4. 3 and 9

5. 5 and 6

6. 8 and 2

7. a and b **8.** x^2 and x **9.** ab and b^2

10. $2a^2$ and $12b^2$ **11.** $2a$ and $3a$ **12.** $6x$ and $3x$

13. $6p$ and $4p$ **14.** $2x$ and $3y$ **15.** ab^2 and a^2b

Example. $6a^2$, $15a$, and 18

To find the LCM of three polynomials, find the LCM of two of them, and then find the LCM of the result and the third.

$$6a^2 = 3 \cdot 2 \cdot a \cdot a \qquad 15a = 3 \cdot 5 \cdot a$$

LCM of $6a^2$ and $15a$ is $3a \cdot 2 \cdot a \cdot 5$, or $30a^2$.

$$30a^2 = 3a \cdot 2 \cdot a \cdot 5 \qquad 18 = 3 \cdot 3 \cdot 2$$

Notice that the only factor of 18 not already in $30a^2$ is the second 3. Thus, the LCM of $30a^2$ and 18 is $3a \cdot 2 \cdot a \cdot 5 \cdot 3$, or $90a^2$.

16. 9, 2, and 6 **17.** 8, 4, and 3 **18.** 5, 2, and 4

19. 5, 4, and 10 **20.** 12, 2, and 3 **21.** 16, 12, and 3

22. 5, 9, and 3 **23.** 7, 4, and 2 **24.** 6, 7, and 3

25. x^2, y, and y^2 **26.** x, y, and z **27.** x^2, xy, and y^2

B Find the LCM of the following.

28. $(x - y)$ and $(x + y)$ **29.** $(x^2 - y^2)$ and $(x - y)$

30. $(4a - 8b)$ and $(3a - 6b)$ **31.** $(9a^2 - 6)$ and $(15a^2 - 10)$

32. $(a + 4)$ and $(a + 2)$ **33.** $(2x + 2y)$ and $(x + y)$

34. $(x + 4)$ and $(x^2 - 16)$ **35.** $(x^2 - 4)$ and $(x + 2)^2$

36. $(x^2 - 5x + 6)$ and $(x^2 - 4x + 3)$

37. $(6a^2 - 5a - 6)$ and $(12a^2 + 11a + 2)$

38. $(8a^2 - 8a - 16)$ and $(10a^2 - 14a - 12)$

9–6 Addition and Subtraction of Rational Expressions

How do you find $\frac{2}{3} + \frac{5}{6}$? How do you find $\frac{5}{8} - \frac{1}{2}$?

You have learned that to add or subtract rational numbers with unlike denominators, you rename the rational numbers so that they have like denominators. That denominator is called the **least common denominator** (LCD).

The same principle is true in algebra. If the denominators of the rational expressions are unlike polynomials, you find the least common multiple of the polynomials, or the least common denominator of the rational expressions. Then proceed as in arithmetic to make the denominators alike, using the following theorem that was proved on page 389.

Theorem 9–1 The numerator and denominator of a rational expression may be multiplied or divided by the same nonzero polynomial without changing the value of the expression.

Study the following parallel arithmetic and algebraic examples.

Arithmetic	*Algebra*
A. Add $\frac{5}{8}$ and $\frac{1}{8}$.	**A.** Add $\frac{x}{x+y}$ and $\frac{3x}{x+y}$.
$\frac{5}{8} + \frac{1}{8} = \frac{5+1}{8} = \frac{6}{8}$	$\frac{x}{x+y} + \frac{3x}{x+y} = \frac{x+3x}{x+y} = \frac{4x}{x+y}$
B. Add $\frac{2}{3}$ and $\frac{4}{5}$.	**B.** Add $\frac{a}{3b}$ and $\frac{c}{2d}$.
① LCD is 15.	① LCD is $6bd$.
② $\frac{2}{3} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{10}{15}$ $\frac{4}{5} = \frac{4 \cdot 3}{5 \cdot 3} = \frac{12}{15}$	② $\frac{a}{3b} = \frac{a \cdot 2d}{3b \cdot 2d} = \frac{2ad}{6bd}$ $\frac{c}{2d} = \frac{c \cdot 3b}{2d \cdot 3b} = \frac{3bc}{6bd}$
③ $\frac{10}{15} + \frac{12}{15} = \frac{22}{15}$, or $1\frac{7}{15}$	③ $\frac{2ad}{6bd} + \frac{3bc}{6bd} = \frac{2ad + 3bc}{6bd}$
C. From $\frac{1\frac{1}{2}}{8}$, subtract $\frac{\frac{1}{4}}{12}$.	**C.** From $\frac{2a+b}{4ab}$, subtract $\frac{3a-2b}{6a}$.
① LCD is 24.	① LCD is $12ab$.
② $\frac{1\frac{1}{2}}{8} = \frac{1\frac{1}{2} \cdot 3}{8 \cdot 3} = \frac{4\frac{1}{2}}{24}$ $\frac{\frac{1}{4}}{12} = \frac{\frac{1}{4} \cdot 2}{12 \cdot 2} = \frac{\frac{1}{2}}{24}$	② $\frac{2a+b}{4ab} = \frac{(2a+b) \cdot 3}{(4ab) \cdot 3} = \frac{6a+3b}{12ab}$ $\frac{3a-2b}{6a} = \frac{(3a-2b) \cdot 2b}{6a \cdot 2b} = \frac{6ab-4b^2}{12ab}$
③ $\frac{4\frac{1}{2}}{24} - \frac{\frac{1}{2}}{24} = \frac{4\frac{1}{2} - \frac{1}{2}}{24}$ $= \frac{4}{24}$, or $\frac{1}{6}$	③ $\frac{6a+3b}{12ab} - \frac{6ab-4b^2}{12ab}$ $= \frac{6a+3b-(6ab-4b^2)}{12ab}$ $= \frac{6a+3b-6ab+4b^2}{12ab}$

Note in Example C that the minus sign affects the whole numerator, $6ab - 4b^2$, and all the signs in it must be changed.

EXAMPLE 1. Subtract $\frac{3x-y}{b}$ from $x+2y$.

① Since the denominator for $(x+2y)$ is 1, the LCD is b.

② $$\left(\frac{x+2y}{1}\cdot\frac{b}{b}\right)-\frac{3x-y}{b}=\frac{bx+2by}{b}-\frac{3x-y}{b}$$
$$=\frac{bx+2by-(3x-y)}{b}$$
$$=\frac{bx+2by-3x+y}{b}$$

EXAMPLE 2. Combine $\frac{3}{x^2}$, $\frac{5}{2xy}$, and $-\frac{4}{3y^2}$.

① The LCD of x^2, $2xy$, and $3y^2$ is $3\cdot 2\cdot x\cdot x\cdot y\cdot y$, or $6x^2y^2$.

② $$\frac{3}{x^2}+\frac{5}{2xy}-\frac{4}{3y^2}=\frac{3\cdot 6y^2}{x^2\cdot 6y^2}+\frac{5\cdot 3xy}{2xy\cdot 3xy}-\frac{4\cdot 2x^2}{3y^2\cdot 2x^2}$$
$$=\frac{18y^2}{6x^2y^2}+\frac{15xy}{6x^2y^2}-\frac{8x^2}{6x^2y^2}$$
$$=\frac{18y^2+15xy-8x^2}{6x^2y^2}$$

EXAMPLE 3. $\frac{x+1}{x^2-9}+\frac{4}{x+3}-\frac{x-1}{x-3}$

① The factors of x^2-9 are $(x+3)$ and $(x-3)$, which are the other two denominators. So the LCD is $(x+3)(x-3)$.

② $$\frac{x+1}{(x+3)(x-3)}+\left(\frac{4}{x+3}\cdot\frac{x-3}{x-3}\right)-\left(\frac{x-1}{x-3}\cdot\frac{x+3}{x+3}\right)$$
$$=\frac{x+1}{x^2-9}+\frac{4x-12}{x^2-9}-\frac{x^2+2x-3}{x^2-9}$$
$$=\frac{x+1+4x-12-(x^2+2x-3)}{x^2-9}$$
$$=\frac{5x-11-x^2-2x+3}{x^2-9},\text{ or }\frac{-x^2+3x-8}{x^2-9}$$

Now try these

— Find the LCD for the following expressions and rewrite them using the LCD.

1. $\frac{2a}{3}, \frac{7a}{5}$
2. $\frac{x}{2y^2}, \frac{3y}{x}$
3. $\frac{y+1}{x}, \frac{2y}{x-1}$

— Perform the indicated operations.

4. $\frac{4}{x}+\frac{6}{y}$
5. $\frac{a}{2b}-\frac{3c}{b^2d}$
6. $\frac{5}{x}-\frac{2}{x+1}+\frac{7}{y}$

Exercises

A — Rewrite the following rational expressions, using the LCD.

1. $\frac{6x}{3}$ and $\frac{5x}{6}$
2. $\frac{x-2y}{6}$ and $\frac{2x-y}{7}$
3. $\frac{a}{a+b}$ and $\frac{b}{a-b}$
4. $\frac{2}{3x^2}$ and $\frac{3}{4xy}$
5. $\frac{x}{yz}, \frac{y}{xz}$, and $\frac{z}{xy}$
6. $\frac{1}{16-x^2}$ and $\frac{9}{4+x}$

— Find the value of each of the following if a is 3 and b is 2.

7. $\frac{1}{a}+\frac{1}{b}$
8. $\frac{3a}{a-b}-\frac{a+b}{a-b}$
9. $\frac{a+b}{b}-\frac{a+b}{a}$
10. $\frac{a}{a^2-b}-\frac{b}{b^2-a}$
11. $\frac{a-b}{a+b}-\frac{a+b}{a-b}$
12. $\frac{a^2}{a-1}+\frac{b^2}{a+1}$

13. Complete the following table by replacing each ? with the sum obtained by adding the rational expression in each row to the rational expression in each column.

	Column: $+$	$\frac{a}{b}$	$\frac{c}{d}$	$\frac{x}{y}$
Row	$\frac{a}{b}$	?	$\frac{ad+bc}{bd}$	?
	$\frac{c}{d}$	$\frac{bc+ad}{bd}$	?	?
	$\frac{x}{y}$	?	?	$\frac{2x}{y}$

— In Exercises 14–17, indicate the letter of one of the following statements that justifies the result.

A. $\frac{P}{Q}=\frac{RP}{RQ}$, $R\neq 0$
B. $\frac{P}{R}+\frac{Q}{R}=\frac{P+Q}{R}$
C. $\frac{P}{Q}+\frac{R}{S}=\frac{PS+QR}{QS}$
D. $\frac{P}{Q}-\frac{R}{S}=\frac{PS-QR}{QS}$

14. $\frac{x+y}{a}+\frac{x}{a}=\frac{2x+y}{a}$
15. $\frac{3a-2b}{4}=\frac{-3a+2b}{-4}$
16. $\frac{5b}{3}-\frac{3b}{8}=\frac{31b}{24}$
17. $\frac{a}{b}+\frac{c}{d}+\frac{e}{f}=\frac{adf+bcf+bde}{bdf}$

— Perform the operations indicated. Simplify your answer when possible.

18. $\frac{x^3}{7}+\frac{x^3}{6}$
19. $\frac{2x}{3}+\frac{5y}{2}$
20. $\frac{x^2y}{6}+\frac{xy^2}{5}$
21. $\frac{7ab}{10}-\frac{3a}{4}$
22. $\frac{2a+3}{6}-\frac{5a-7}{9}$
23. $\frac{3x-5}{4}+\frac{5x-3}{3}$
24. $\frac{4n-1}{5}-\frac{n+2}{4}$
25. $\frac{x^2-y^2}{3}-\frac{x^2+y^2}{8}$
26. $\frac{3c}{8}+\frac{b+c}{3}$
27. $\frac{1}{a}+\frac{1}{b}$
28. $\frac{2}{a}+\frac{3}{b}$
29. $\frac{1}{a}-\frac{1}{b}$
30. $\frac{4}{x}-\frac{5}{y}$
31. $\frac{a}{b}-\frac{c}{d}$
32. $\frac{3}{2c}+\frac{4}{6c}$
33. $\frac{5}{4a}-\frac{3}{8a}$
34. $3+\frac{1}{a}$
35. $ax+\frac{b}{x}$
36. $3+\frac{a}{b}$
37. $2x-\frac{x+y}{y}$
38. $\frac{7a}{10p}-\frac{2b}{5p}$
39. $\frac{5r}{4c}+\frac{4s}{5d}$
40. $\frac{3}{x}+\frac{5}{x^2}$
41. $\frac{4a}{b^2}-\frac{3a}{b}$
42. $\frac{3}{x}-\frac{5}{x^3}+\frac{2}{x^2}$
43. $\frac{a}{x^3}+\frac{b}{x^2}-\frac{c}{x}$
44. $\frac{5}{ab^2}-\frac{7}{a^2b}$
45. $a-\frac{b}{c}$
46. $\frac{a}{x}-(a-1)$
47. $a^2-\frac{1}{a}$
48. $\frac{2}{xy}+\frac{3}{yz}$
49. $\frac{9}{mn}+\frac{3}{mn^2}$
50. $\frac{8}{x}+\frac{3}{xy}$
51. $\frac{1}{6p}-\frac{1}{4p}+\frac{1}{3p}$
52. $\frac{3}{b}+\frac{5}{2b}-\frac{11}{3b}$
53. $\frac{a+b}{b}-\frac{a-b}{a}$
54. $\frac{x+1}{2x}+\frac{2}{x}$
55. $v+\frac{m_1v_1}{m}$
56. $\frac{2x-1}{x}-\frac{x+3}{3x}$
57. $\frac{1}{2a^2}-\frac{5}{6ab}+\frac{7}{12b^2}$
58. $\frac{x}{2b}+\frac{2x}{b}-\frac{1}{b^2}$
59. $\frac{x-1}{3xy}+\frac{x^2-x}{9x^2y^2}$
60. $\frac{2a+1}{a}+\frac{3b+5}{b}$
61. $\frac{3a+5}{a}-\frac{2b-3}{b}$
62. $\frac{3x-y}{x}+\frac{4x+2y}{x}$
63. $\frac{4x+y}{x}-\frac{3x-4y}{x^2}$
64. $\frac{2y-3}{2y}+\frac{y-4}{3y}$
65. $\frac{5a-2b}{-a^2}+\frac{4a+3b}{-3a^2}$
66. $\frac{6r-2s}{-2s^2}-\frac{5r+2s}{s^2}$

B — Combine as the signs indicate.

67. $\frac{5}{x+2}+\frac{3}{x-2}$
68. $\frac{5}{x+5}-\frac{3}{x-5}$
69. $\frac{2}{a+3}+\frac{5}{a+5}$
70. $\frac{2x}{x-y}-\frac{3y}{x+y}$
71. $\frac{a}{2a+2b}-\frac{b}{3a+3b}$
72. $\frac{4a}{6a-2b}+\frac{3b}{9a-3b}$
73. $\frac{2}{3t+3s}+\frac{3}{5r-5s}$
74. $\frac{3x}{2y-3}-\frac{2x}{3y-2}$
75. $\frac{x+3}{x-5}+\frac{x-5}{x+3}$
76. $\frac{a-2}{a+3}-\frac{a-3}{a+5}$
77. $\frac{x+3}{x^2-4}+\frac{x-5}{x+2}$
78. $\frac{3a+2}{3a+6}-\frac{a-2}{a^2-4}$
79. $\frac{a+b}{ax+ay}-\frac{a+b}{bx+by}$
80. $\frac{a-b}{a+b}-\frac{a}{a^2-b^2}$
81. $\frac{x-y}{x+y}+\frac{4xy}{x^2-y^2}$
82. $\frac{x+y}{x-y}-\frac{4xy}{x^2-y^2}$

C — Perform the operation indicated.

83. $\frac{4a}{2a+6}-\frac{a-1}{a+3}$
84. $\frac{x-6y}{2x^2+5xy+2y^2}-\frac{7}{x+2y}$
85. $\frac{2m-6}{2m^2-10m+12}-\frac{5}{m-3}$
86. $\frac{7}{a^2+a-2}-\frac{5}{a^2-4a+3}$
87. $\frac{2x}{x^2-25}-\frac{4(x-5)}{x-5}$
88. $\frac{3}{y^2+y-6}+\frac{2}{y^2-4y+4}$
89. $\frac{5}{a^2-2ab+b^2}+\frac{6}{a-b}$
90. $\frac{5}{3x-3}+\frac{x}{2x+2}-\frac{3x^2}{x^2-1}$
91. $\frac{x^2}{a^2-b^2}+\frac{x}{(a-b)^2}$
92. $\frac{x^2}{x+y}-\frac{y^2}{x-y}-(x-y)$

9–7 The Signs in a Rational Expression

Any rational number, such as $\frac{4}{5}$, shown without a sign is understood to be positive. Thus, you can think of $\frac{4}{5}$ as meaning $+\frac{+4}{+5}$. Three signs are associated with it.

Any rational expression has three signs.

① The sign of the rational expression.
② The sign of the numerator of the rational expression.
③ The sign of the denominator of the rational expression.

The three signs of the expression $-\frac{2x}{y^2}$ are

① $-$ for the rational expression, $\frac{2x}{y^2}$.
② $+$, understood, for the numerator, $2x$.
③ $+$, understood, for the denominator, y^2.

What are the three signs of $\frac{-2x}{y^2}$?

Study the following examples. Notice that, although the expressions have the same value, the signs differ.

a. $+\frac{+6x^2}{+2x} = +3x$

b. $+\frac{-6x^2}{-2x} = \frac{-1}{-1}\left(+\frac{-6x^2}{-2x}\right) = +\frac{+6x^2}{+2x} = +3x$

c. $-\frac{-6x^2}{+2x} = -1\left(-\frac{(-1)(-6x^2)}{+2x}\right) = +\frac{+6x^2}{+2x} = +3x$

d. $-\frac{+6x^2}{-2x} = -1\left(-\frac{+6x^2}{(-1)(-2x)}\right) = +\frac{+6x^2}{+2x} = +3x$

Since each rational expression equals $+3x$, they are all equal to each other.

$$+\frac{+6x^2}{+2x} = +\frac{-6x^2}{-2x} = -\frac{-6x^2}{+2x} = -\frac{+6x^2}{-2x}$$

From the examples, you can see that by introducing -1 twice as a factor in the coefficients, an expression may be changed to an equivalent expression.

Theorem 9–5 The factor (-1) may be introduced twice into any rational expression without changing the value of the expression.

First prove that $\frac{P}{Q} = -\frac{-P}{Q}$ is true.

Statements	*Reasons*
1. $\frac{P}{Q} = \frac{P}{Q}$	1. Reflexive property.
2. $= \frac{-1}{-1} \cdot \frac{P}{Q}$	2. Theorem 9–1.
3. $= \frac{-P}{(-1)Q}$	3. Postulate of multiplication.
4. $= \frac{(1)}{(-1)} \cdot \frac{-P}{Q}$	4. Postulate of multiplication.
5. $= -1\left(\frac{-P}{Q}\right)$, or $-\frac{-P}{Q}$	5. Definition of division of real numbers.

Thus, $\frac{P}{Q} = -\frac{-P}{Q}$ is true. Two of the signs have been changed.

Similarly prove that $\frac{P}{Q} = -\frac{P}{-Q}$ and $\frac{P}{Q} = \frac{-P}{-Q}$ are true.

The final statement for the entire proof would be that

$$\frac{P}{Q} = -\frac{-P}{Q} = -\frac{P}{-Q} = \frac{-P}{-Q} \text{ is true.}$$

What would the reason be?

This theorem is often helpful in combining rational expressions.

EXAMPLE. Combine $\frac{3}{x-y} - \frac{2}{y-x}$.

If $(y-x)$ were multiplied by -1, it would become $(-y+x)$, or $x-y$, and the rational expressions would have the same denominator. Remember that one more sign of the rational expression must be changed.

$$-\frac{2}{y-x} = -1\left(-\frac{2}{(-1)y-x}\right) = +\frac{2}{-(y-x)}$$
$$= \frac{2}{-y+x}, \text{ or } \frac{2}{x-y}$$

Notice that the $-$ sign in the denominator applies to the entire expression $y-x$. By removing the parentheses, $-(y-x)$ becomes $-y+x$, or $x-y$.

Now combine the rational expressions.

$$\frac{3}{x-y} + \frac{2}{x-y} = \frac{5}{x-y}$$

Could you have chosen to change the expression $\frac{3}{x-y}$ so that its denominator would be $(y-x)$ and then combine? Try it.

When you change the sign of a numerator or denominator that is a polynomial, be sure to distribute the (-1) factor over the entire polynomial. Thus, $\frac{-1}{-1} \cdot \frac{-2x^2-3x+1}{-3x+2}$ becomes $\frac{2x^2+3x-1}{3x-2}$.

When you change the sign of a numerator or denominator that is made up of factors, distribute the (-1) factor over *one* factor only.

$$\frac{-1}{-1} \cdot \frac{(a-b)(c+d)}{a(b-e)} = \frac{(-a+b)(c+d)}{a(-b+e)}$$

Changing more than one factor in either the numerator or denominator would be the same as multiplying by more than *one* (-1) factor, and that does not agree with the theorem. Which of the following expressions is the same as $\frac{(x-3)(x+2)}{5(x^2+y^2)}$?

a. $\frac{(x-3)(-x-2)}{-5(x^2+y^2)}$ b. $\frac{(x-3)(-x-2)}{-5(-x^2-y^2)}$

Now try these

— Tell whether or not the following pairs of rational expressions are equal.

1. $\frac{-ax^2}{by^3} = -\frac{ax^2}{by^3}$
2. $\frac{3+r}{s+2} = \frac{-3+r}{-s+2}$
3. $-\frac{x^2-9}{y^2+4} = \frac{-(x^2-9)}{y^2+4}$
4. $\frac{5xy}{-9y^2} = \frac{-5xy}{9y^2}$

Answers: 1. Equal. 2. Not equal. 3. Equal. 4. Equal.

Exercises

A — If the following pairs of rational expressions are equal, write *Equal*. If they are not equal, rewrite the second expression so it does equal the first.

1. $\frac{a}{b}$ and $\frac{-a}{-b}$
2. $\frac{3}{x^2}$ and $-\frac{-3}{x^2}$
3. $\frac{y}{9}$ and $\frac{-y}{9}$
4. $\frac{a+b}{-7}$ and $-\frac{a+b}{7}$
5. $\frac{x^2+y}{x}$ and $-\frac{x^2+y}{x}$
6. $\frac{ab+1}{a}$ and $\frac{ab+1}{-a}$
7. $\frac{3}{a-b}$ and $\frac{-3}{b-a}$
8. $\frac{c+d}{a}$ and $\frac{d-c}{a}$
9. $\frac{x}{(a-3)(a+5)}$ and $-\frac{x}{15-2a-a^2}$
10. $\frac{a+b}{a-b}$ and $\frac{a+b}{b-a}$

— Combine as indicated.

11. $\frac{a}{a-b} - \frac{b}{b-a}$
12. $\frac{5}{x^2} + \frac{7}{-x^2}$
13. $\frac{4}{x-2} + \frac{2}{2-x}$
14. $\frac{y}{y-3} + \frac{1}{3-y}$
15. $\frac{10}{2y-1} - \frac{6}{1-2y}$
16. $\frac{3a}{x-y} + \frac{2b}{y-x}$

B — Combine as indicated.

17. $\frac{a^2}{a^2-1} + \frac{a}{1-a}$
18. $\frac{4}{x-3} - \frac{x}{9-x^2}$
19. $\frac{a-b}{a+b} - \frac{a-b}{b+a}$
20. $\frac{2a+3b}{2} - \frac{a-b}{4}$

— Simplify these rational expressions.

21. $\frac{a-b}{b-a}$
22. $\frac{a+b}{b+a}$
23. $\frac{a^2-b^2}{b-a}$
24. $\frac{a^2-b^2}{b+a}$
25. $\frac{(x-2)(x-3)}{(3-x)(2-x)}$
26. $\frac{(a-b)^2}{(b-a)^2}$

9–8 Other Rational Expressions

Consider the following rational expressions.

a. $\frac{\frac{7}{4}}{\frac{3}{4}}$ b. $\frac{\frac{3}{4}}{\frac{7}{8}}$ c. $\frac{\frac{a}{b}-1}{c}$ d. $\frac{x}{\frac{x}{y}}$ e. $\frac{c-\frac{b}{a}}{b-\frac{a}{b}}$

There are two methods to write each of examples **a–e** as a rational expression in a simple form, that is, in a form without a rational expression $\frac{P}{Q}$ in its numerator or its denominator.

METHOD I: Make the numerator a single rational expression and the denominator a single rational expression. Then divide the single rational expressions.

$$\frac{3-\frac{x}{y}}{\frac{2x}{y}} = \frac{\frac{3y-x}{y}}{\frac{2x}{y}} = \frac{3y-x}{y} \cdot \frac{\cancel{y}}{2x} = \frac{3y-x}{2x}$$

METHOD II: Multiply the numerator and the denominator by a number or expression so that you will not have a rational expression in either the numerator or the denominator. That number or expression will be the LCD of all rational expressions in the numerators and denominators of the complex rational expressions.

$$\frac{3-\frac{x}{y}}{\frac{2x}{y}} = \frac{y}{y} \cdot \frac{\left(\frac{3}{1}-\frac{x}{y}\right)}{\left(\frac{2x}{y}\right)} = \frac{3y-x}{2x}$$

Now try these

— Simplify the following expressions. Do Exercises 1–4 by Method I and Exercises 5 and 6 by Method II.

1. $\frac{\frac{x}{4}}{3}$
2. $\frac{b}{\frac{a}{b}}$
3. $\frac{\frac{x^2}{y^2}}{\frac{x}{y}}$
4. $\frac{\frac{x+y}{x}}{\frac{x^2-y^2}{x^2y}}$
5. $\frac{a+\frac{b}{c}}{b+\frac{c}{d}}$
6. $\frac{1-\frac{x}{y}}{\frac{1}{x}-\frac{1}{y}}$

Answers: 1. $\frac{x}{12}$ 2. $\frac{b^2}{a}$ 3. $\frac{x}{y}$ 4. $\frac{xy}{x-y}$ 5. $\frac{acd+bd}{bcd+c^2}$ 6. x

Exercises

A — Use either method to simplify the following rational expressions.

1. $\dfrac{\frac{1}{2}}{4}$
2. $\dfrac{20}{\frac{1}{4}}$
3. $\dfrac{\frac{2}{3}}{1\frac{1}{2}}$
4. $\dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{2}+\frac{1}{3}}$
5. $\dfrac{\frac{a}{b}+\frac{c}{d}}{ac}$
6. $\dfrac{\frac{a}{b}-\frac{c}{d}}{ad}$
7. $\dfrac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{xy}}$
8. $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{ab}}$
9. $\dfrac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b}}$
10. $\dfrac{\frac{x}{y}-\frac{y}{x}}{x+y}$
11. $\dfrac{x+y}{\frac{1}{x}+\frac{1}{y}}$
12. $\dfrac{\frac{1}{y}-\frac{1}{x}}{x^2-y^2}$
13. $\dfrac{\frac{1}{x}-1}{x-x^2}$
14. $\dfrac{\frac{1}{y}-1}{\frac{1}{y}+1}$
15. $\dfrac{\frac{a}{b}-1}{a^2-b^2}$
16. $\dfrac{1-\frac{x^2}{y^2}}{x+y}$
17. $\dfrac{x+\frac{1}{x}}{x}$
18. $\dfrac{x}{x+\frac{1}{x}}$
19. $\dfrac{\frac{1}{x}-x}{x}$
20. $\dfrac{\frac{a^2-b^2}{2}}{a+b}$
21. $\dfrac{\frac{a}{b}-\frac{b}{a}}{a-b}$
22. $\dfrac{x-\frac{9}{x}}{x+3}$
23. $\dfrac{9-\frac{b}{y}}{-\frac{3}{y}}$
24. $\dfrac{-8a-\frac{7}{b}}{-\frac{9}{b}}$

B — Simplify the following.

25. $\dfrac{4xy-\frac{y}{2}+y}{-\frac{y}{3}}$
26. $\dfrac{\frac{12x^2y}{9z}}{\frac{4xy}{3z}}$
27. $\dfrac{\frac{36rs^2}{7r^2s^2}}{\frac{4rs}{21r^2s}}$
28. $\dfrac{\frac{a-b}{c}}{\frac{a+b}{3c}}$
29. $\dfrac{\frac{a+2}{2a}}{\frac{a-2}{a}}$
30. $\dfrac{x^2-25}{x-5}$
31. $\dfrac{\frac{x+2}{x-2}}{\frac{2x+4}{x^2-4}}$
32. $\dfrac{\frac{a}{b}-1}{\frac{a}{b}+1}$
33. $\dfrac{\frac{y}{x^2}+x}{xy+\frac{1}{x}}$

C — Simplify the following rational expressions.

34. $\dfrac{\frac{2}{x-1}+\frac{x-1}{x+1}}{1+\frac{1}{x}}$
35. $\dfrac{\frac{a-4}{a+4}-1}{\frac{a+4}{a-4}+\frac{2}{a+4}}$
36. $\dfrac{\frac{x-y}{x^2+y^2}-\frac{1}{x}}{\frac{x-y}{x^2+y^2}-\frac{1}{y}}$
37. $1+\dfrac{1}{x+\dfrac{1}{x+\frac{1}{x}}}$

More Challenging Problems

— Perform the indicated operations and simplify the results.

1. $\dfrac{x+y+\frac{y^2}{x}}{x+y+\frac{x^2}{y}}$
2. $1-\dfrac{m(m-n)}{1-mn}$
3. $\dfrac{x+\frac{y^3}{x^2}}{\frac{y^2}{x^2}-\frac{y}{x}+1}$
4. $\dfrac{\frac{x^2}{y^2}-1}{\frac{x^2}{y^2}+\frac{2x}{y}+1}$
5. $\dfrac{mx+my-2x-2y}{mx-my-2x+2y}$
6. $\dfrac{x^3+y^3}{4x^2+xy-3y^2}$
7. $\dfrac{\frac{a^3-b^3}{a^2b-a^2}}{\frac{2a}{b^2-a^2}}\cdot\dfrac{2a}{a^2-b^2}$
8. $\dfrac{\frac{3^{n2+1}}{3^{(n+2)^2}}}{\frac{3^{2(n-1)}}{3(n)^0}}$
9. $\dfrac{1-x^3}{1+\dfrac{x}{1-\frac{x}{1+x}}}$
10. $\dfrac{2x+2}{x+1-\dfrac{x^2+x+1}{x+\frac{1}{x-1}}}$
11. $\dfrac{\dfrac{1-4x^2}{(2x+y)^2}\left[1+\dfrac{y+1}{2x-1}\right]}{\dfrac{1}{2x+y}-\dfrac{1}{2x-y}+\dfrac{1}{4x^2-y^2}}$
12. $\dfrac{x-6+\frac{4}{x-2}}{\left(\frac{1}{x+2}-\frac{x}{7x-4}\right)\left(x-\frac{x+8}{x-1}\right)}$
13. $\dfrac{\frac{1}{a^2}-\frac{1}{4d^2}}{\frac{1}{a}+\frac{1}{2d}}-\dfrac{\frac{1}{a^3}-\frac{1}{8b^3}}{\frac{1}{a^2}-\frac{1}{4b^2}}$
14. $1-\dfrac{2}{x+\dfrac{3}{x-\frac{4}{x}}}$

Find $\dfrac{x-y}{x+y}$ in terms of a and b if $x=\dfrac{a}{b+a}$ and $y=\dfrac{a-b}{-a}$.

16. $\frac{4x^2+8}{x^2+2x-3} - \frac{6x+1}{x+3} = \frac{2x-7}{1-x}$

17. $\frac{15}{2x+1} - \frac{5}{4x^2-1} + \frac{6}{2x-1} = 0$

18. $\frac{1+2x}{x^2-4x-21} = \frac{1+x}{6+5x+x^2} + \frac{x+5}{x^2-5x-14}$

19. A storekeeper paid a dollars to two men, paying n times as many dollars to one as the other. How much did he pay each man in terms of a and n?

CHAPTER REVIEW

1. Know the meaning of and be able to use each of the following words or phrases. The number shown after each word or phrase indicates where it is introduced, in case you need to review.

rational expression (*386*)
simplifying a rational expression (*390*)
least common multiple (LCM) (*398*)
least common denominator (LCD) (*400*)
signs of a rational expression (*405*)

2. If $\frac{A}{B} = \frac{C}{D}$ ($B, D \neq 0$), which of the following statements is true?

a. $AC = BD$ b. $A + C = B + D$
c. $AD = BC$ d. $A + D = B + C$

3. By what process do you check the equality of two rational expressions?

Tell if the following pairs of expressions are *Equal* or *Unequal*.

4. $\frac{x-5}{x}, \frac{x^2-10x+25}{x-5}$

5. $\frac{10x^2-13x-3}{2x-3}, \frac{5ax+a}{a}$

6. $\frac{14x^2-x-3}{2x^2-7x+3}, \frac{7x+3}{x-3}$

Write the following expressions in simplest form.

7. $\frac{3a(x^2-4y^2)}{a(x-2y)}$

8. $\frac{5x^2-38x-63}{5x^2+2x-7}$

9. State the theorem that permits you to write this product.

$$\frac{x-3}{x+4} \cdot \frac{x-2}{x+1} = \frac{x^2-5x+6}{x^2+5x+4}$$

10. What is the product of $\frac{7nx^3}{3n^2y^3}$ and $\frac{15y^2}{21x^2}$ in simplest form?

Supply the correct rational expression to make each of the following sentences true.

11. $\frac{3}{a} + \frac{5}{b} = \frac{3}{a} \cdot \frac{?}{}$

12. $\frac{4b^2}{-13ab} \div \frac{2b}{a} = \frac{4b^2}{-13ab} \cdot \frac{?}{}$

13. What is the quotient in Exercise 11? in Exercise 12?

Perform the indicated operation. Write your answers in simplest form.

14. $\frac{7x-y}{-5x^2} - \frac{x-3y}{3x^2}$

15. $\frac{-2a}{4a-b} + \frac{3a}{\frac{b}{2}-2a}$

16. $\frac{\frac{x-y}{xy}}{x^2-y^2} \cdot xy$

17. $\frac{-3+\frac{2}{a}}{\frac{a^2}{-3a+2}}$

18. Show that $\frac{x-y}{y-x}$ and -1 are equivalent.

19. The length of a rectangle is $3n + 4$. The width is $\frac{1}{3}$ the length. Write expressions for the perimeter and the area of the rectangle.

CHAPTER TEST

Replace ● with $=$ or $\neq$ to make true statements.

1. $\frac{x-1}{y} \bullet \frac{x}{y}$

2. $\frac{x^2y}{x^3y} \bullet \frac{1}{x}$

3. $\frac{ax+ay}{bx+by} \bullet \frac{a(x+y)}{b(x+y)}$

4. $\frac{p^2-3p-4}{2p^2-14p+24} \bullet \frac{p+1}{p-3}$

5. $\frac{17+t^2}{-x^2y} \bullet -\frac{17+t^2}{x^2y}$

6. $\frac{5g^2h^3}{17i} \bullet -\frac{-5g^2h^3}{-17i}$

7. $\frac{2m^2+mn}{4m^2n} \bullet \frac{2m+n}{4mn}$

8. $\frac{16a^2b^2c}{12a^3+6a} \bullet -\frac{-8b^2c}{6a^2+3}$

Perform the operations indicated. Simplify your answers.

9. $\frac{a^2-36}{10ab} \cdot \frac{2b}{a+b}$

10. $\frac{4a^4b^4}{a^2-b^2} \div \frac{2a^3b^3}{a^2-2ab+b^2}$

11. $\frac{2x-3}{4x+5} + \frac{2x-5}{4x+5}$

12. $a - \frac{a+b}{b}$

13. $\dfrac{a-\frac{1}{b}}{a+\frac{1}{b}}$

14. $\dfrac{\frac{a}{b}-\frac{b}{a}}{\frac{a}{b}-1}$

15. $\dfrac{a}{a-2}-\dfrac{1}{a^2-4}$

16. $\dfrac{4}{x^2-5x+6}-\dfrac{5}{x^2+2x-15}$

17. $a-\dfrac{2}{a-2}+1$

18. $\dfrac{3a+5}{a}-\dfrac{3b-2}{b}$

19. $\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)\left(\dfrac{3a^2b}{b-a}\right)$

20. $\dfrac{x^2-5x-6}{x^3}\cdot\dfrac{x}{x+1}$

CUMULATIVE REVIEW

Give the letter of the response that best completes the sentence or answers the question.

1. What is the value of $\frac{2a+b}{-3c}$ if a is -4, b is 2, and c is -1?

a. -2 b. $+2$ c. -6 d. $+6$

2. Which equation is represented graphically by a line with a slope of $-\frac{7}{4}$, passing through (0, 3)?

a. $4y-7x=12$ b. $-4y+7x=12$
c. $4y+7x=12$ d. $4y+7x=-12$

3. If the cost of automobile tires is \$15 each, the cost, n, of any number of tires, t, may be represented as $n=15t$. What is a sensible replacement set for t?

a. The rational numbers. b. The real numbers.
c. The integers. d. The natural numbers.

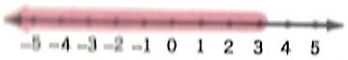
4. Given $\frac{a}{b}$ and $\frac{c}{d}$ with $a, c \in W$ and $b, d \in N$, which sentence is false?

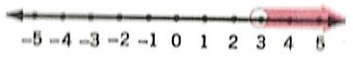
a. $\frac{a}{b}=\frac{c}{d}$ if $ad=bc$. b. $\frac{a}{b}>\frac{c}{d}$ if $ad>bc$.
c. $\frac{a}{b}>\frac{c}{d}$ if $bc>ad$. d. $\frac{a}{b}<\frac{c}{d}$ if $ad<bc$.

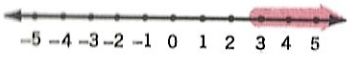
5. If the replacement set is the set of integers, what is the solution set of $3x-2>4$?

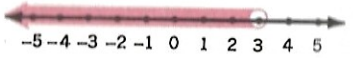
a. $\{2, 3, 4, 5, \cdots\}$ b. $\{3, 4, 5, 6, \cdots\}$
c. $\{1, 2, 3, 4, \cdots\}$ d. $\{-3, -4, -5, -6, \cdots\}$

6. Which graph correctly represents the solution set of $7x-3(x+5)<-3$? The replacement set is R.

a. 

b. 

c. 

d. 

7. Which postulate for the real numbers is illustrated by $3a^2b+7a^2b=10a^2b$?

a. Commutative. b. Distributive.
c. Associative. d. Additive identity.

8. Which of the following sentences is equivalent to $8-5x=-7x+2$?

a. $-12x+10=0$ b. $2x=-3$
c. $2x=-6$ d. $\frac{7}{5}x=6$

9. What is the solution set of $9x-3=5(3x-1)$?

a. $\{-\frac{1}{3}\}$ b. $\{\frac{1}{3}\}$ c. $\{3\}$ d. $\{-3\}$

10. What is the slope of the line that passes through the two points $(-6, 2)$ and $(3, -5)$?

a. $-\frac{7}{9}$ b. $\frac{7}{9}$ c. $\frac{9}{7}$ d. $-\frac{9}{7}$

11. Which number pair correctly completes the following?

$$\{(x, y) : 3x-5y=13\} \cap \{(x, y) : -x-3y=5\} = \{(\underline{?}, \underline{?})\}$$

a. $(-1, 2)$ b. $(-1, -2)$ c. $(1, -2)$ d. $(-2, 1)$

12. By what postulate are these two sentences equivalent?

$$(3n+10)+5n-10 \quad \text{and} \quad 3n+(10+5n)-10$$

a. Commutative. b. Associative.
c. Distributive. d. Transitive.

13. Which exponent theorem is illustrated by $3x^{\frac{1}{2}}\cdot 5x^{\frac{3}{4}}=15x^{\frac{5}{4}}$?

a. $(xy)^n=x^ny^n$ b. $x^{\frac{1}{n}}=\sqrt[n]{x}$
c. $(x^m)^n=x^{mn}$ d. $x^m\cdot x^n=x^{m+n}$

14. To rationalize the denominator of $\dfrac{x^3}{x-\sqrt{2}}$, multiply numerator and denominator by $\underline{?}$.

a. $x-\sqrt{2}$ b. $x+\sqrt{2}$ c. $\sqrt{2}$ d. $-\sqrt{2}$

15. Consider the following system of equations.

$$\begin{cases} -15x + 2y = -7 \\ 7x - 10y = 1 \end{cases}$$

The graph of which of the following equations passes through the intersection of the graphs of the equations in the system?

a. $x = -\frac{1}{2}$ **b.** $y = -\frac{1}{4}$
c. $2x + 1 = 0$ **d.** $2x - 1 = 0$

16. Which term makes $x^2 + 1\frac{1}{3}x$ a perfect square trinomial?

a. $\frac{2}{3}$ **b.** $-\frac{4}{3}$ **c.** $\frac{4}{9}$ **d.** $-\frac{4}{9}$

17. What are the correct factors of the polynomial $6a^3 - 9a^2b - 6ab^2$?

a. $3(a - 2b)(2a + b)$ **b.** $3a(a - 2b)(2a + b)$
c. $3a(a + 2b)(2a - b)$ **d.** $-3(a - 2b)(2a + b)$

18. What are the zeros of the function $y = 12x^2 + 17x - 5$?

a. $\frac{1}{4}$ and $\frac{5}{3}$ **b.** $\frac{1}{4}$ and $-\frac{5}{3}$
c. $-\frac{1}{4}$ and $\frac{5}{3}$ **d.** $-\frac{1}{4}$ and $-\frac{5}{3}$

19. What is the quotient of $x^2 - 9x + 8$ divided by $x - 1$?

a. $x + 9$ **b.** $x - 9$ **c.** $x - 8$ **d.** $x + 8$

20. Which of the following is equivalent to $\frac{a}{-b}$?

a. $-\frac{a}{b}$ **b.** $-\frac{a}{-b}$ **c.** $\frac{-a}{-b}$ **d.** $-\frac{-a}{b}$

ENRICHMENT

Continued Fractions

How are the following two expressions different?

a. $1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}}$

b. $2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{4 + \cdots}}}}}}$

Both **a** and **b** are **continued fractions** but **a** is a *terminating* continued fraction, since it has a definite last term. In this section the primary topic is simple terminating, continued fractions, generally

mbolized as follows, where each a_i after a_1 must be an integer, and numerators after a_1 are equal to 1.

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots + \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}}}$$

You may ask "Is a continued fraction such as **a** just invented or is it arrived at in a particular way? Does it have a particular value?" To determine its value, start at the bottom and add upward.

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{\frac{21}{5}}}} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{5}{21}}}$$

$$= 1 + \cfrac{1}{2 + \frac{1}{\frac{68}{21}}} = 1 + \cfrac{1}{2 + \frac{21}{68}} = 1 + \cfrac{1}{\frac{157}{68}} = 1 + \frac{68}{157} = \frac{225}{157}$$

Then **a** is the rational number $\frac{225}{157}$. Can $\frac{225}{157}$ be expanded back to a continued fraction? Begin by dividing 225 by 157.

$$\frac{225}{157} = 1 + \frac{68}{157}$$

You can make $\frac{68}{157}$ a rational expression with a numerator of 1 by making its multiplicative inverse the denominator.

$$\frac{225}{157} = 1 + \frac{68}{157} = 1 + \cfrac{1}{\frac{157}{68}}$$

Now you can divide 157 by 68 and continue the process.

$$1 + \frac{1}{\frac{157}{68}} = 1 + \frac{1}{2 + \frac{21}{68}}, \text{ or } 1 + \cfrac{1}{2 + \cfrac{1}{\frac{68}{21}}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{5}{21}}}, \text{ or } 1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{\frac{21}{5}}}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}}$$

Can you go any further with the expansion of $\frac{22}{157}$? If you express $\frac{1}{5}$ as $\frac{1}{\frac{5}{1}}$, you obtain $\frac{1}{5}$ again, so the process is ended. In general, when you reach a fraction that has a numerator of 1, the continued fraction has terminated.

Although the proof will not be given here, a theorem and its converse can be stated.

Theorem If n can be written as a simple terminating, continued fraction, then n can be written as a rational number.

The converse would read "If n can be written as a rational number, then n can be written as a simple terminating, continued fraction."

Exercises

1. Express these continued fractions as rational numbers.

a. $3 + \cfrac{1}{5 + \cfrac{1}{7 + \frac{1}{9}}}$

b. $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}}$

c. $0 + \cfrac{1}{6 + \cfrac{1}{4 + \frac{1}{2}}}$

2. Expand these rational numbers as continued fractions.

a. $\frac{11}{9}$

b. $\frac{23}{17}$

c. $\frac{11}{15}$ (*Hint:* Start with $a_1 = 0$; that is, $\frac{11}{15} = 0 + \frac{11}{15} = 0 + \cfrac{1}{\frac{15}{11}} = 0 + \cfrac{1}{1 + \frac{4}{11}}$, and so forth.)

d. $-\frac{15}{7}$ (*Hint:* $-\frac{15}{7} = -3 + \frac{6}{7}$; now expand $\frac{6}{7}$ as shown in **c** and add.)

3. a. Express $\frac{83}{65}$ as a continued fraction.

b. Multiply both numerator and denominator of $\frac{83}{65}$ by 2 to obtain the equivalent fraction $\frac{166}{130}$. Express $\frac{166}{130}$ as a continued fraction.

c. Multiply $\frac{83}{65}$ by $\frac{3}{3}$ and express the equivalent fraction, $\frac{249}{195}$, as a continued fraction.

d. What do you notice about the three continued fractions for **a**, **b**, and **c**? If you add up each of the continued fractions, what one fraction will you obtain?

e. If you have done your computation correctly, you have illustrated a very interesting property of continued fractions. State it in your own words.

THE GROWTH OF MATHEMATICS

Algebra *Is an Arabic Word*

With the rapid rise of the Islam religion, starting in A.D. 622, the conquering Arabs soon came to occupy most of the intellectual centers of the ancient world. And because of their religion, they were faced with a number of problems, such as the calculation of the dates of religious festivals and the times for daily prayer, that required mathematics. Arab rulers answered the need by having translations made of the classical Greek works of mathematics. The Arabs were also in contact with India, so they were soon in possession of the flexible decimal place-value numeration system with the all-important zero. Thus, Arab mathematicians were in an excellent position to synthesize the known mathematical learning and to extend it.

The writer who was most successful at this synthesis was Muhammad ibn-Musa al-Khwarizmi (A.D. 780–?850). The Latin form of his name, *algorism*, came to mean "arithmetic" in Europe. Today, *algorism* denotes any definite procedure for solving a mathematical problem. Al-Khwarizmi's best-known book, *Restoration and Reduction*, was written in 830. When translated in the twelfth century, it reintroduced the work of Diophantus and introduced the work of the Indian writers on algebra to Europe. The word *algebra* derives from the Arabic title of this book, *al-jebr w'almuquabala*.

Many other Arabic writers contributed to arithmetic and algebra, but not extensively to geometry. They often gave geometric illustrations to explain why the algebraic operations were performed; however, they did not use proofs.

Some Arabic writers used negative numbers. They were able to solve all equations involving the square of a variable, but they did not know how to solve equations involving the cube of a variable. The main contribution of the Arabs was the preservation of the Greek works and the introduction of Indian mathematics to Europe.

CHAPTER 10

USING RATIONAL EXPRESSIONS

You know what a rational expression is, and now you will learn some of the many ways rational expressions are used in mathematics.

● Have you ever whirled a bucket of water around in a circle as the boy in the upper left photograph is doing? If you have, you should have found that the water did not spill out, even when the container was upside-down. This is because objects moving in circular paths tend to move away from the center of their motion. This tendency, called centrifugal force, can be measured by the formula $F = \frac{Mv^2}{gr}$. In this formula, F is the centrifugal force, M is the mass in pounds, v is the velocity in feet per second, g is a constant, 32.2 feet per second per second, and r is the radius of the circle in feet. Much work in science and business is aided by such rational expressions.

● If you were a traffic controller in a railroad yard, you would want to be sure that the two trains in the lower left photograph were on different tracks before they reached each other. Finding the time when they would meet can be solved by rational expressions.

● If an engineer had wanted to know before he started drilling how long the tunnel through the redwood would be, he would not have been able to measure it directly. If the tree trunk were circular, how would knowing its circumference help him to find the distance through the trunk? In this chapter, you will learn other methods of indirect measurement that involve triangles.

● The instrument being used in the upper right photograph is called a transit. It measures angles and enables the surveyor to calculate distances and other data. If you know two sides of a right triangle, how would you find the third, using methods that you already know? There are relationships between the acute angles and the sides of a right triangle that you will learn to use to make indirect measurements.

To solve $\frac{x}{2} + \frac{2x}{3} = \frac{5}{4}$, first multiply each term by the LCD, 12. Then you can easily solve the simplified equation for x.

$$12\left(\frac{x}{2}\right) + 12\left(\frac{2x}{3}\right) = 12\left(\frac{5}{4}\right)$$
$$6x + 8x = 15$$

In general, to solve equations with rational expressions, the procedure is the same. Multiply each term by the LCD and solve for the variable. Study the following examples.

EXAMPLE 1. Solve $\frac{x}{x+6} = 13$.

① The LCD is $x + 6$.

② Multiply each term by $x + 6$.

③ Solve for x.

$$\frac{x}{x+6} = 13$$
$$(x+6)\left(\frac{x}{x+6}\right) = (x+6)(13)$$
$$x = 13x + 78$$
$$-12x = 78$$
$$x = -6\tfrac{1}{2}$$

Check
$$\frac{-6\frac{1}{2}}{-6\frac{1}{2}+6} = 13$$
$$\frac{-6\frac{1}{2}}{-\frac{1}{2}} = 13$$
$$(-\tfrac{13}{2})\cdot(-\tfrac{2}{1}) = 13$$
$$13 = 13$$

EXAMPLE 2. Solve $\frac{2}{3x} + \frac{1}{2} = \frac{3}{4x}$.

① The LCD of $3x$, 2, and $4x$ is $12x$.

② Multiply each term by $12x$.

$$\frac{2}{3x} + \frac{1}{2} = \frac{3}{4x}$$
$$12x\left(\frac{2}{3x}\right) + 12x\left(\frac{1}{2}\right) = 12x\left(\frac{3}{4x}\right)$$
$$(4\cdot 2) + (6x\cdot 1) = (3\cdot 3)$$
$$8 + 6x = 9$$
$$6x = 1$$
$$x = \tfrac{1}{6}$$

Check
$$\frac{2}{3(\frac{1}{6})} + \tfrac{1}{2} = \frac{3}{4(\frac{1}{6})}$$
$$\frac{2}{\frac{1}{2}} + \tfrac{1}{2} = \frac{3}{\frac{2}{3}}$$
$$(2\cdot\tfrac{2}{1}) + \tfrac{1}{2} = (3\cdot\tfrac{3}{2})$$
$$4 + \tfrac{1}{2} = \tfrac{9}{2}, \text{ or } 4\tfrac{1}{2}$$

EXAMPLE 3. Solve $\frac{7}{x-4} = \frac{5}{x+2}$.

① The LCD is $(x-4)(x+2)$.

② Multiplying each term by the LCD gives the following equation.

$$(x+2)(7) = (x-4)(5)$$

③ Solve for x.

$$7x + 14 = 5x - 20$$
$$2x = -34$$
$$x = -17$$

④ Check to be sure.

Now try these

Solve.

1. $\frac{15}{2x} = 3$
2. $\frac{8}{x+2} = 8$
3. $\frac{a}{a+5} = \frac{1}{2}$
4. $\frac{4}{3x} + \frac{6}{5x} + \frac{3}{10} = \frac{1}{3}$
5. $\frac{2}{x+3} = \frac{4}{x+1}$
6. $\frac{x-4}{7} + \frac{2}{x-2} = 1$

Answers: 1. $2\frac{1}{2}$ 2. -1 3. 5 4. 76 5. -5 6. 9 or 4

Exercises

A Solve the following equations.

1. $\frac{2}{x} = 3$
2. $\frac{3}{a} = -4$
3. $\frac{2}{3x} = \frac{1}{9}$
4. $\frac{3}{2x} - \frac{1}{3} = \frac{5}{6x}$
5. $\frac{5}{6x} + 3 = \frac{1}{2x}$
6. $\frac{3}{10x} - \frac{3}{5} = \frac{2}{5x}$
7. $\frac{5}{n} - \frac{1}{2} = 2$
8. $\frac{4}{x} + \frac{3}{2x} = \frac{11}{6}$
9. $\frac{2n+1}{2n} - \frac{3n-2}{3n} = \frac{7}{12}$
10. $\frac{4}{2x} - 3 = \frac{-2}{5x} - \frac{3}{5}$
11. $\frac{4}{3x} - \frac{5}{2x} = 5 - \frac{1}{6x}$
12. $\frac{7}{4x} - 2 = \frac{3}{2x} - 4$
13. $\frac{3}{2} = \frac{x}{x+2}$
14. $\frac{b-2}{b+3} = \frac{3}{8}$
15. $\frac{x-7}{x+2} = \frac{1}{4}$
16. $\frac{5}{a+3} = \frac{2}{2}$
17. $\frac{a-1}{a-2} = 1.5$
18. $\frac{y}{y-3} = \frac{6}{3}$
19. $\frac{2}{x+3} = \frac{5}{x}$
20. $\frac{-3}{x+5} = \frac{1}{2}$

21. $\frac{x}{3} + \frac{2x}{3x-4} = \frac{9x-2}{9}$

22. $\frac{2x-1}{2} - \frac{x+2}{2x+5} = \frac{6x-5}{6}$

23. $\frac{7}{x-4} = \frac{5}{x-2}$

24. $\frac{10}{x-3} = \frac{9}{x-5}$

25. $\frac{x+5}{x-3} + \frac{4}{x-3} = 5$

26. $\frac{x+2}{x+4} = \frac{x+6}{x-8}$

27. $\frac{x-1}{x+1} = \frac{x+3}{x+10}$

28. $\frac{x-3}{x+5} = \frac{x-2}{x+2}$

C — Solve the following.

29. $\frac{3}{3x^2-3x-28} = \frac{5}{5x^2-x-20}$

30. $\frac{x}{x^2-9} < 4 - \frac{4x}{x-3}$

31. $\frac{2x+1}{x-1} - \frac{3x}{x+2} = \frac{-x^2-2}{x^2+x-2}$

32. $\frac{a}{a-2} = \frac{a+3}{a+2} - \frac{a}{a^2-4}$

33. $\frac{3(x-4)}{4x^2-9} = \frac{3x}{4x^2-16x+15}$

34. $\frac{1}{1-x^2} + \frac{1}{x-1} > 0$

35. $\frac{7x}{x^2-2x-8} = \frac{3}{x-4} + \frac{4x}{x^2-5x+4}$

10–2 Problems with Rational Expressions

Some word problems can be solved by using one equation with one variable. Other problems are easier to solve by using two equations with two variables. And many problems can be worked by either method. By reading the conditions of the problem, you should be able to decide whether it is easier to express the given conditions in terms of one variable or in terms of two variables.

EXAMPLE 1. What number must be subtracted from the numerator and added to the denominator of $\frac{17}{22}$ to make a rational number equal to $\frac{5}{8}$?

The problem says "what number." State the condition in terms of one variable.

① Let x = unknown number.
$17 - x$ = new numerator.
$22 + x$ = new denominator.

② Then write the equation stated in the problem.

$$\frac{17-x}{22+x} = \frac{5}{8}$$

③ Solve for x.

$$8(22+x)\left(\frac{17-x}{22+x}\right) = 8(22+x)\left(\frac{5}{8}\right)$$
$$8(17-x) = (22+x)(5)$$
$$136 - 8x = 110 + 5x$$
$$-13x = -26$$
$$x = 2$$

Check $\frac{17-2}{22+2} \stackrel{?}{=} \frac{5}{8}$

$\frac{15}{24} = \frac{5}{8}$

When solving a problem using two equations with two variables, each of the two conditions is expressed in an equation. This gives two equations that can be solved by either the addition method or the substitution method that you learned earlier.

EXAMPLE 2. Separate 120 into two parts so that one part is $\frac{3}{5}$ of the other part. (To separate a number into two parts means to find two addends whose sum is that number.)

① Let x be *one part*,
y be *the other part.*

② Use the two conditions of the problem to write two equations.

I Separate 120 into two parts. $x + y = 120$
II One part is $\frac{3}{5}$ of the other part. $x = \frac{3}{5}y$

③ Choose a method to solve the problem. Since x is given as $\frac{3}{5}y$, use the substitution method.

I $x + y = 120$
$\frac{3}{5}y + y = 120$
$\frac{8}{5}y = 120$
$8y = 600$
$y = 75$ *the other part*

II $x = \frac{3}{5} \cdot 75$
$= 45$ *one part*

④ Check in the problem.

Exercises

A

1. What number must be added to the numerator and subtracted from the denominator of the rational number $\frac{17}{32}$ to make a rational number equal to $\frac{3}{4}$?

2. The numerator of a rational number is 3 smaller than its denominator; if 2 is added to the numerator, the value of the rational number becomes $\frac{3}{4}$. What is the rational number?

3. The denominator of a rational number is 7 more than its numerator. If the numerator is diminished by 6, the resulting rational number equals $\frac{1}{2}$. What was the original rational number?

4. Separate 95 into two parts so that one part is $\frac{3}{7}$ of the other.

5. One number is 4 more than another. The quotient of the larger divided by the smaller is $\frac{5}{2}$. Find the numbers.

6. One number is 7 less than another. The quotient of the larger divided by the smaller is $\frac{4}{3}$. Find the numbers.

7. Separate 72 into two parts so that their quotient is $\frac{2}{3}$.

8. What number added to both the numerator and the denominator of $\frac{3}{5}$ will make the value of the resulting rational number $\frac{3}{4}$?

9. The sum of two rational numbers is $\frac{11}{16}$, and one of them is $\frac{5}{6}$ of the other. Find the rational numbers.

10. The difference between two rational numbers is $\frac{1}{8}$. One rational number is $\frac{2}{3}$ of the other. Find the rational numbers.

11. Two integers are consecutive (n and $n+1$). If 6 is added to the first and 2 is subtracted from the second, the quotient of the resulting integers is $4\frac{1}{2}$. Find the integers.

12. The numerator of a rational number exceeds its denominator by 3. If 1 is added to the denominator and 5 is subtracted from the numerator, the value of the rational number becomes $\frac{1}{2}$. What is the rational number?

13. The numerator of a rational number is 8 less than the denominator. If 3 is added to the numerator and 1 is subtracted from the denominator, the resulting rational number is $\frac{3}{5}$. What is the rational number?

14. The perimeter of a rectangle is 52 inches. How long is each side if the altitude is 2 inches more than $\frac{1}{3}$ the base?

15. Find three consecutive integers such that $\frac{1}{2}$ the first plus $\frac{2}{3}$ the second less $\frac{3}{4}$ the third will be 10.

B

16. A is $\frac{5}{6}$ as old as B. Five years ago, A was $\frac{4}{5}$ as old as B was then. How old is A now? How old is B now?

17. The sum of the angles of any triangle equals 180°. If in a certain triangle, $\frac{3}{5}$ of the first angle equals 34° less than the second and $\frac{5}{6}$ of the first angle equals the third, how many degrees are there in each angle?

18. Susan is now $\frac{4}{5}$ as old as her older sister Jean. Four years ago she was $\frac{3}{4}$ as old as her sister. What are their present ages?

19. Gail is now 24 years younger than her mother. Eight years ago she was $\frac{1}{4}$ as old as her mother. What are their present ages?

20. A seed store has 300 pounds of lawn-seed mixture that is $\frac{3}{4}$ rye grass and $\frac{1}{4}$ blue grass. How much blue grass must be added to make a mixture that is $\frac{1}{2}$ rye grass and $\frac{1}{2}$ blue grass? (*Hint:* The amount of rye grass remains the same.)

21. Mr. Zack has $\frac{1}{3}$ of his savings invested at 5%, $\frac{1}{4}$ at 4%, and the rest at 6%. The total interest per year on these investments is $372. How much does he have invested at each rate?

10–3 Work Problems

Work problems usually involve persons or machines working together at different rates of speed. The method for solving these problems involves deciding how much of the work an individual can do in a unit of time, such as a minute, an hour, or a day.

The following examples should help you.

EXAMPLE 1. Arthur can shovel snow from a sidewalk in 60 minutes and Jack can do it in 30 minutes. How long will it take them to do the job together?

① Let n = number of minutes for both to do the job together.

② Arthur can do $\frac{1}{60}$ of the job in 1 minute and
$\frac{n}{60}$ of the job in n minutes.
Jack can do $\frac{1}{30}$ of the job in 1 minute and
$\frac{n}{30}$ of the job in n minutes.

③ The sum of the parts of the job that each does in n minutes equals 1, the whole job.

$$\begin{array}{lcl} \text{part done by Arthur} + \text{part done by Jack} & = & \text{whole job} \\ \frac{n}{60} + \frac{n}{30} & = & 1 \end{array}$$

④ Multiply by the LCD, 60, and solve for n.

$$\begin{aligned} 60\left(\frac{n}{60}\right) + 60\left(\frac{n}{30}\right) &= 60(1) \\ n + 2n &= 60 \\ 3n &= 60 \\ n &= 20 \end{aligned}$$

⑤ They can do the job together in 20 minutes. Check the solution.

EXAMPLE 2. Jeff can do a certain job in 8 days. After working alone for 4 days, he is joined by Charlie and together they finish the work in 2 more days. How long would it take Charlie alone?

① If Jeff can do the job alone in 8 days, then he can do
$\frac{1}{8}$ of the job in 1 day and
$\frac{6}{8}$, or $\frac{3}{4}$, of the job in 6 days.

② Let n be the number of days it would take Charlie to do the job alone. Then
$\frac{1}{n}$ is the part of the job he can do in 1 day and
$\frac{2}{n}$ is the part of the job he can do in the 2 days he works with Jeff.

③ The sum of the parts done by Jeff and Charlie equals one whole job.

$$\frac{3}{4} + \frac{2}{n} = 1$$

④ Solve for n.

$$4n\left(\frac{3}{4}\right) + 4n\left(\frac{2}{n}\right) = 4n(1)$$
$$3n + 8 = 4n$$
$$-n = -8$$
$$n = 8$$

⑤ Check the solution with the conditions of the problem to be sure you are correct.

Exercises

A

1. One man can make a sidewalk in 4 days, and another can do it in 4.5 days. How long will it take them working together?

2. A large pipe can empty a tank in 5 minutes, and a smaller pipe can empty it in 8 minutes. How long would it take to empty the tank if both pipes were draining the tank?

3. Mike can do a piece of work in 3 days, Ted in 5 days, and Kevin in 8 days. If all three work together, how long will it take them to do the work?

4. To remove the dirt for a building foundation, a contractor is using a large diesel shovel and a small one. If the large one can do the whole job in 6 days and the small one can do it in 9 days, how long will it take them together?

5. Mr. Hall estimates he can paint Mrs. Smith's house alone in 6 days. One of his helpers could paint it alone in 8 days and the other helper could paint it alone in 10 days. How long will it take all three working together?

6. One machine can complete an order for bolts in 7 hours and another machine can do it in 5 hours. How long will it take both machines to finish the job after the slower machine has been working alone for $3\frac{1}{2}$ hours?

B

7. How long will it take Mary and Alice to perform a job together if Mary takes 12 days alone and Alice takes 8 days?

8. It takes 4 minutes to fill a certain bathtub and 8 minutes to empty it when it is full. With the drain open and the tub empty, how long will it take to fill the tub?

10–4 Motion Problems

If you have ever rowed a boat, you know it seems easier to row and the boat goes faster if you go with the current, rather than against it.

Similarly, pilots of aircraft are glad to pick up a tailwind, a wind from behind the plane, since they can complete their flight in less time. The speed of the plane with respect to the ground (the ground speed) is the usual air speed plus the speed of the wind. On the other hand, a headwind, a wind at the front of the plane, cuts the speed.

The following example involves varying rates of motion.

EXAMPLE. The air speed of an airplane is 225 miles per hour. Flying from city A to city B, it has a tailwind of 25 miles per hour. It takes 3 hours longer to fly from B to A than from A to B. How far is it between the two cities?

① Let n = distance between A and B.
250 mph = speed from A to B.
200 mph = speed from B to A.
Then $\frac{n}{250}$ = number of hours to fly from A to B,
$\frac{n}{200}$ = number of hours to fly from B to A.

② Since it takes 3 hours longer to fly from B to A than from A to B, the following equation is true.

$$\frac{n}{200} - \frac{n}{250} = 3$$

(Continued on next page.)

③ Multiply by the LCD, 1000, and solve for n.

$$1000\left(\frac{n}{200}\right) - 1000\left(\frac{n}{250}\right) = 1000(3)$$
$$5n - 4n = 3000$$
$$n = 3000 \text{ miles}$$

Exercises

A

1. Jake can row at the speed of 6 mph in still water. In a river where the current is 3 mph, it takes him 4 hours longer to row x miles upstream than the same distance downstream. Find x.

2. Tim can average 12 mph with his motor boat in still water. In a river with a current of 4 mph, it takes him 9 hours to travel from point A to point B and return. Find the distance from A to B.

3. On a trip between two cities, an airplane took $3\frac{1}{2}$ hours flying at a speed of 360 mph with the wind. The return trip between the cities took $4\frac{1}{2}$ hours and the speed and direction of the wind was the same. Find the speed of the airplane on the return trip. What was the speed of the wind?

4. James drove his car 300 miles in the same time that Edgar drove his car 270 miles. James drove 5 miles an hour faster than Edgar. Find the speed of each.

5. An automobile traveled $\frac{1}{5}$ as fast as an airplane. The airplane required $\frac{1}{2}$ hour less time to go 900 miles than the automobile required to go 210 miles. Find the speed of each.

6. A motorist drove 120 miles at a certain rate. Driving back, he doubled his rate, and took 3 hours less time. Find his rate going.

7. Allan can drive his car over a route in 5 hours, and John can drive his car over the same route in 4 hours. How long would it take them to meet if they started at opposite ends at the same time?

Racetrack Puzzle

Two bike riders, Robinson and Brown, start at the same time and race on a circular track. Robinson can circle the track in 6 minutes and Brown in 4 minutes. From the beginning of the race, how many minutes will it be before Brown overtakes Robinson?

10–5 Solving Rational Equations for One of the Variables

On a loan of \$500 for one year, Mr. Frederick pays \$30 interest. He wants to figure out the rate of interest he pays. He knows the formula for finding the interest, $prt = i$, and uses that to obtain a new formula for the rate of interest.

Both sides of the equation are divided by pt so r appears in terms of the other three variables.

Then, this new formula can be used to find the rate of interest.

$$prt = i$$
$$\frac{prt}{pt} = \frac{i}{pt}$$
$$r = \frac{i}{pt}$$
$$r = \frac{\$30}{\$500 \times 1}, \text{ or } \frac{30}{500}$$
$$r = \frac{6}{100}, \text{ or } 6\%$$

Many of the formulas used by scientists, engineers, and businessmen contain rational expressions. For instance, in physics, you would use the formula

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

to find the combined resistance of three electric circuits in parallel connection. Or you might use the formula

$$V = \tfrac{1}{6}h(B + b + 4M)$$

to find the volume of a prismatoid. Even if you do not completely understand the formulas, you can solve for any variable; and if you were given values for the variable, you could use the formula.

You should notice that some formulas contain such letters as R and r or B and b. The capital and lower-case letters are different variables, just as a and b or x and y are different variables. The same is true when **subscripts** are used to differentiate between variables; for example, C_L, C_a, r_1, and r_2 all represent different quantities. (These are read "C sub L," "C sub a," "r sub 1," "r sub 2.")

The following examples show how to change the form of an equation by solving for one variable in terms of the others, just as $prt = i$ was solved for r.

EXAMPLE 1. Solve for a in the equation $s(a + b) = c$.

① Use distributive property.
② Add $-sb$.
③ Multiply by $\frac{1}{s}$.

$$s(a + b) = c$$
$$sa + sb = c$$
$$sa = c - sb$$
$$a = \frac{c - sb}{s}$$

EXAMPLE 2. Solve for n in the equation $\frac{n+2}{c} = \frac{n+c}{b}$.

① Multiply by bc.
$$bc\left(\frac{n+2}{c}\right) = bc\left(\frac{n+c}{b}\right)$$
$$b(n+2) = c(n+c)$$
$$bn + 2b = cn + c^2$$

② Get terms with n on one side.
$$bn - cn = c^2 - 2b$$

③ Factor to have n by itself.
$$n(b-c) = c^2 - 2b$$

④ Solve for n.
$$n = \frac{c^2 - 2b}{b - c}$$

Now try these

— Solve for n.

1. $na + n = cd$ **2.** $\frac{n}{x} + 1 = x - d$ **3.** $\frac{n-a}{b} = \frac{n}{2}$

Answers: **1.** $n = \frac{cd}{a+1}$ **2.** $n = x^2 - dx - x$ **3.** $n = \frac{-2a}{b-2}$, or $\frac{2a}{2-b}$

Exercises

A — Solve for the required variable in terms of the other variables.

1. $i = prt$ Solve for r.

2. $s = \frac{W}{L}$ Solve for L.

3. $M = \frac{bh^3}{3}$ Solve for b.

4. $v = \frac{S}{T}$ Solve for T.

5. $s = \frac{ah}{r}$ Solve for a.

6. $v = \frac{bh^3}{3}$ Solve for h.

7. $I = \frac{bd^3}{a}$ Solve for b.

8. $A = \frac{F}{M}$ Solve for M.

9. $A = \frac{1}{2}bh$ is the formula for finding the area of a triangle. Solve it for b.

10. $I = \frac{E}{R}$ is called Ohm's Law and is used to determine electric current flowing through a circuit. Find R.

11. $F = \frac{Mv^2}{gr}$ is the formula used to compute centrifugal force, the kind of force you feel when a car goes around a curve. Solve for g.

12. $K = \frac{1}{2}mv^2$ is used to find kinetic energy, the energy of a moving body. Solve for m.

13. $V = \frac{4}{3}\pi r^3$ is used to find the volume of a sphere. Solve for π.

14. $S = \frac{1}{2}gt^2$ enables you to compute the distance an object falls in time t. Solve for g.

15. $A = p + prt$ is used by bankers to calculate your total savings principal, plus interest. Solve for p.

— Solve for n.

16. $2n + a = b$

17. $2n - a = b$

18. $an - b = c$

19. $n(a - b) = c$

20. $n(a + b) + c = d$

21. $an - bn = c$

22. $an - bn + cn = d$

23. $an + bn + c = d$

24. $\frac{1}{3}n + a = b$

25. $\frac{1}{3}n + a = n$

26. $\frac{n}{a} = b$

27. $\frac{n}{a} = \frac{b}{c}$

B — Solve for n.

28. $\frac{n}{a} + \frac{b}{c} = d$

29. $\frac{a}{c} = n + bn$

30. $\frac{n}{a+b} = c$

31. $\frac{n}{a+b} = c - n$

32. $\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$

33. $\frac{1}{n} = \frac{1}{a} - \frac{1}{b}$

34. $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$ is used to determine the focal length of a lens. Solve for f.

35. If you wanted to find the sum of an infinite number of terms in a sequence such as $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$, you could use the formula $S = \frac{a}{1-r}$. Solve this for r.

36. If you wanted to find a certain term of the sequence 5, 7, 9, 11, $\cdots$, you would use the formula $l = a + (n-1)d$. Solve for n.

37. What is the sum of the first one hundred even integers? The formula $S = \frac{n}{2}(a + e)$ would aid you in finding the sum. Express the formula in terms of a.

38. $M = \frac{25}{f} + \frac{25}{d}$ is used to determine the magnifying power of a simple microscope. (25 is 25 centimeters, the distance for the most distinct image.) Express this formula in terms of d.

— Solve for the required variables.

39. $r = \frac{v^2 pL}{a}$ Solve for p and for a.

40. $R = \frac{WL - x}{L}$ Solve for W, for L, and for x.

41. $f = \frac{gm - t}{m}$ Solve for g, for m, and for t.

42. $S = \frac{rl - a}{r - l}$ Solve for r, for l, and for a.

C **43.** Assume that all variables in the formula $f = \frac{gm - t}{m}$ represent positive numbers. If the value of m is increased and g and t remain fixed, is the value of f *increased* or *decreased*? What happens if t is negative and m is increased? Give evidence to support your answer.

44. The formula for the coefficient of lift, used in aerodynamics, is $C_L = \frac{L}{\frac{P}{2}SV^2}$. Solve for S. Then find the value of S when $L = 864$, $P = .003$, $C_L = .1$, and $V = 120$.

45. The formula $W = C_L \frac{P}{2} SV^2$ can be changed in order to find the velocity needed to sustain the weight of an airplane under certain conditions. Solve for V^2. Then find V when $W = 1500$, $C_L = .43$, $P = .002$, and $S = 200$.

10–6 Indirect Measurement

One application of rational expressions occurs in the study of **trigonometry.** Although *trigonometry* means "measurement of triangles," its application is extensive and is not limited to measuring triangles.

For now, the next few sections will acquaint you with how trigonometry is used to measure triangles and how triangle relationships can be used to measure distances indirectly.

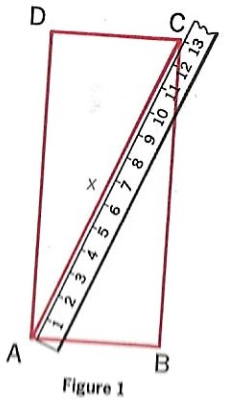


Figure 1

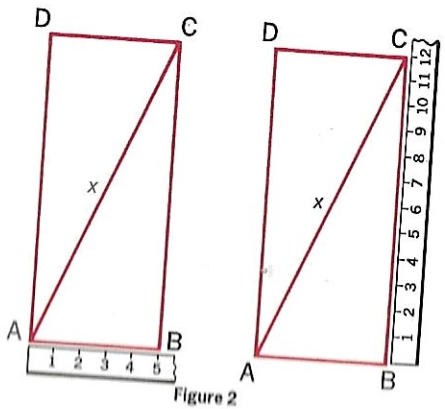


Figure 2

In Figure 1 at the bottom of the previous page, the measure of diagonal AC is found by **directly** measuring the length. The "measure of $\overline{AC}$" is written symbolically as $|\overline{AC}|$.

In Figure 2, $|\overline{AC}|$ is found **indirectly.** Sides AB and BC are measured directly as 5 inches and 12 inches, respectively. Then these lengths are used with the Pythagorean theorem to find the length of the diagonal AC.

$$\begin{aligned} |\overline{AC}|^2 &= |\overline{AB}|^2 + |\overline{BC}|^2 \\ &= (5)^2 + (12)^2 \\ &= 25 + 144, \text{ or } 169 \\ |\overline{AC}| &= \sqrt{169}, \text{ or } 13 \end{aligned}$$

You may want to use indirect measurement either for convenience or out of necessity, for example, to find the length of a bridge to span a river or to find the length of a tunnel to be dug through a mountain.

To make measurements indirectly, you use direct measurements and a known relation between the direct measurements and the unknown.

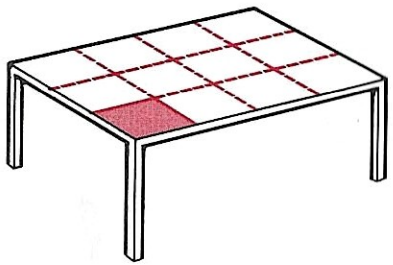
Exercises

A **1.** To find the area of a table top, Tom used a square piece of cardboard one foot on a side, making it one square foot in area. He applied it to the table top and found the number of square units needed to cover the top of the table.

a. Is he measuring directly or indirectly?

b. What is the unit of measurement?

c. How many 1-foot squares could Tom place on the table top?



2. Byron measured the length and width of the table directly with a ruler and then used these lengths to find the area.

a. If the length is measured as 4 feet and the width as 3 feet, find the area using the formula $A = lw$.

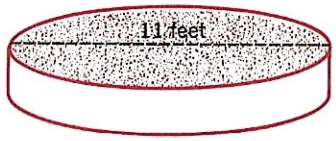
b. In this case, was the area measured directly? Explain.

c. What measurements were made directly?

d. What measurement was obtained indirectly?

3. The diameter of a circular sand box was measured as 11 feet.

a. Write the formula for the relation between the diameter and circumference of a circle.



b. Find the circumference of the sand box.

c. Did you find the circumference directly or indirectly?

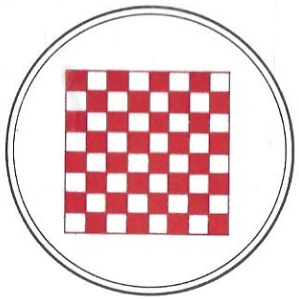
4. It took the Halls 8 hours to make a trip of 375 miles by car.

a. Write the formula showing the relation among time, t, distance, d, and rate, r.

b. Find their average rate of travel.

c. Did you find their average rate of travel directly or indirectly? Explain.

5. Dolores wants to put marble tiles on a circular table top so she can use it as a chess board.



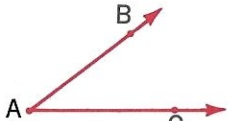
a. The diameter of the table is 14 inches. Using the formula $A = \pi r^2$, find the area in square inches.

b. If the tiles are 1-inch squares, does the number for the area tell you the number of tiles you will need to cover the table top?

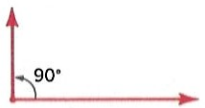
c. If you know that one side of the playing area is 8 inches, how many tiles will you need to cover it? Can you find the answer directly? indirectly? Which is easier?

10–7 Angles and Triangles

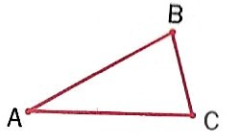
At the right is an **angle,** which is formed by two rays with a common endpoint. This angle is called angle A, which may be written $\angle A$, and is formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Rays AB and AC are the **sides** of the angle, and point A is the **vertex.**



The measure of an angle is determined by comparing it with a unit angle. One unit that is used to measure angles is the degree. With the degree as a unit, it takes 90 unit angles to make a **right angle.**



You know that if you join three points by segments, the figure formed is a triangle. By joining points A, B, and C, you form triangle ABC, written $\triangle ABC$.

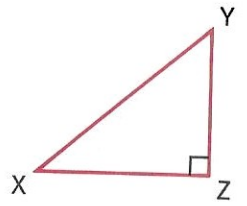


The sum of the measures of the angles of a triangle is 180°. An angle whose measure is less than 90° is an **acute angle.** An angle whose measure is greater than 90° is an **obtuse angle.**

A triangle that has a right angle is a **right triangle.** The right angle is shown by the symbol ⌝ in the triangle. Since the sum of the measures of the angles of a triangle is 180°, the other two angles of a right triangle must be acute angles.

For $\triangle XYZ$ at the right,

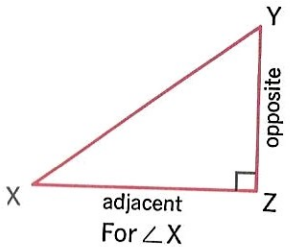
$$|\angle X| + |\angle Y| + |\angle Z| = 180°.$$

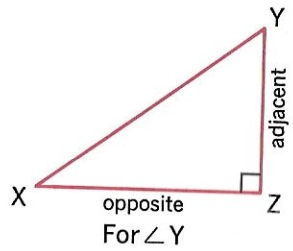


If $|\angle Z|$ is 90° and $|\angle X|$ is 40°, you can find the measure of $\angle Y$ indirectly.

$$\begin{aligned} |\angle X| + |\angle Y| + |\angle Z| &= 180° \\ 40° + |\angle Y| + 90° &= 180° \\ |\angle Y| + 130° &= 180° \\ |\angle Y| &= 50° \end{aligned}$$

In order to distinguish the three sides of a right triangle, they are named as shown in the figure.



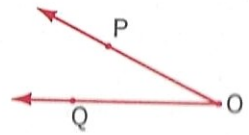


In a right triangle, the side opposite the right angle is called the **hypotenuse** to distinguish it from the *sides.*

Exercises

A

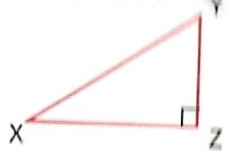
1. Name the vertex of the angle at the right. What are the two sides of the angle?



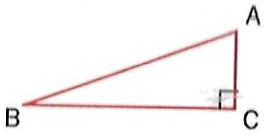
2. If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are each made 4 inches longer, is the measure of $\angle O$ smaller, larger, or the same?

angle and $|\angle X|$ is 35°. What is $|\angle Y|$?

4. One of the angles of a right triangle measures 68°. What is the measure of the other two angles?



5. a. Triangle ABC is a right triangle. If $|\angle A|$ is 70°, what is $|\angle B|$? Is the sum of $|\angle A|$ and $|\angle B|$ equal to 90°?



b. If one acute angle of a right triangle measures 33°, the other acute angle measures ? °. The sum of the measures of the two acute angles is ? °.

c. What is the sum of the measures of the two acute angles in a right triangle?

d. If one acute angle of a right triangle measures x°, the other acute angle measures ? °.

6. Use triangle ABC at the right to complete these statements.

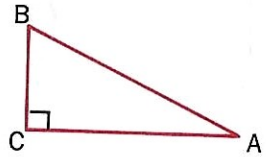
a. The side opposite $\angle A$ is ?.

b. The side adjacent to $\angle A$ is ?.

c. The side opposite $\angle B$ is ?.

d. The side adjacent to $\angle B$ is ?.

e. The hypotenuse is ?.



7. Use right triangle MNP to complete these statements.

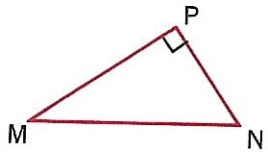
a. The side opposite $\angle M$ is ?.

b. The side adjacent to $\angle M$ is ?.

c. The side opposite $\angle N$ is ?.

d. The side adjacent to $\angle N$ is ?.

e. The hypotenuse is ?.



B **8.** One acute angle of a right triangle measures 10° more than 3 times the measure of the other acute angle. What are the measures of the three angles?

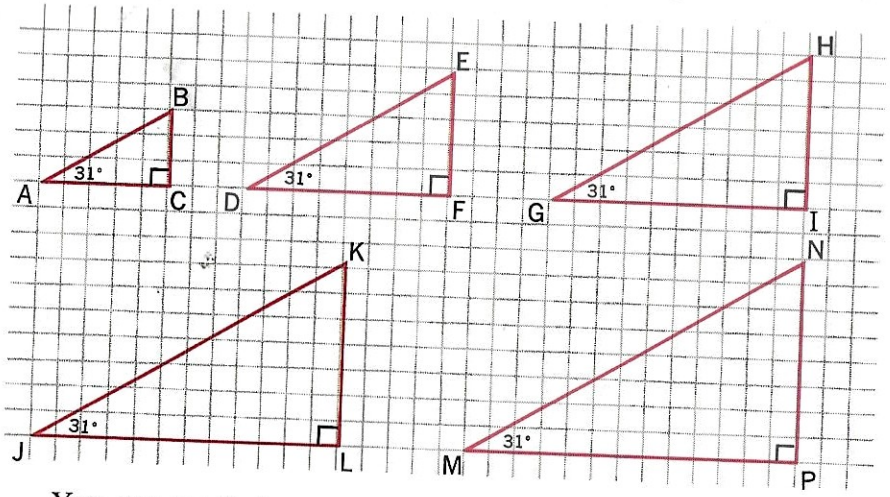
9. In $\triangle ABC$, $|\angle A|$ is 7° less than twice $|\angle B|$, and $|\angle C|$ is 3° more than 5 times $|\angle B|$. What are $|\angle A|$, $|\angle B|$, and $|\angle C|$?

10. The measure of one angle of a triangle is 40° more than $\frac{1}{2}$ the measure of a second angle and 20° less than a third angle. If the first angle measures 60°, find the measures of the other two angles.

The Tangent Ratio

When a rational number $\frac{a}{b}$ is used to represent the division of one number by another, the quotient, $a \div b$, is called the *ratio of a to b*. A **ratio** is the comparison of two numbers by division.

Look at the right triangles below. From the following experiment, you will discover an important idea about right triangles.



You can see that one angle of each triangle is 31° and that you can find the length of each side of the triangles by counting the number of units in that segment.

On a piece of paper, copy the following chart. In the first column, write the length of the side adjacent to the 31° angle. Then find, to the nearest tenth, the ratio of the measures of the two sides. Record your results in the third column. The first one is done for you.

Right triangle	Length of side opposite 31° angle	Length of side adjacent to 31° angle	$\frac{\lvert\text{opposite side}\rvert}{\lvert\text{adjacent side}\rvert}$
ABC	3	5	$\frac{3}{5}$, or .6
DEF			
GHI			
JKL			
MNP			

Look at your results and then answer the following questions.

a. Are the lengths of the sides opposite the 31° angle the same in any two triangles?

in any two triangles?

c. What is the ratio $\frac{|\text{ side opposite } 31^\circ \text{ angle }|}{|\text{ side adjacent to } 31^\circ \text{ angle }|}$? Will this be the same for any 31° angle? Can you explain the results?

If two angles of one triangle are equal in measure to two angles of another triangle, the triangles are **similar.** The ratio of the measures of the corresponding sides of similar triangles is *always the same.*

The five triangles that you used for your ratio experiment each had a 31° angle and a 90° angle. Thus, they are all similar, and the ratio $\frac{|\text{ opposite side }|}{|\text{ adjacent side }|}$ is always the same. This ratio is called the **tangent.**

Definition For any given acute angle of a right triangle, the ratio of the measure of the side opposite the angle to the measure of the side adjacent to the angle is called the tangent.

The abbreviation for *tangent* is **tan.** The tangent of 20° is *tan* 20°. For $\triangle ABC$, the following ratios hold.

$$\tan |\angle A| = \frac{|\overline{BC}|}{|\overline{AC}|}, \text{ or } \frac{a}{b}$$

$$\tan |\angle B| = \frac{|\overline{AC}|}{|\overline{BC}|}, \text{ or } \frac{b}{a}$$

Direct measuring of the lengths of the sides of a triangle and computation of the tangent is time consuming and often produces inaccurate results. Because tangent values are widely used, tables of such values have been prepared for easy reference. The *Table of Sine, Cosine, and Tangent Values* appears on page 547. The values of the tangents of angles from 0° to 89° are computed to four decimal places, as you can see in the portion of the table reproduced at the right. Now study the examples on the following page.

ANGLE	SINE	COSINE	TANGENT
40	.6428	.7660	.8391
41	.6561	.7547	.8693
42	.6691	.7431	.9004
43	.6820	.7314	.9325
44	.6947	.7193	.9657
45	.7071	.7071	1.0000
46	.7193	.6947	1.0355
47	.7314	.6820	1.0724
48	.7431	.6691	1.1106
49	.7547	.6561	1.1504

In the column headed ANGLE, find 41°. Directly opposite 41°, in the column headed TAN, read .8693.

$$\tan 41^\circ \approx .8693$$

Nearly all values in the table are approximations.

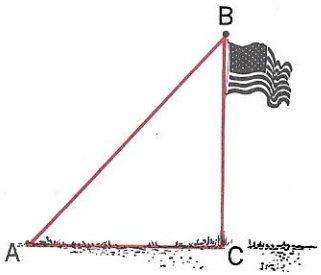
EXAMPLE 2. What is the measure of the angle whose tangent is 1.1106?

In the column headed TAN, find 1.1106. Directly opposite 1.1106, in the column headed ANGLE, read 48°.

$$\tan 48^\circ \approx 1.1106$$

The measure of the angle whose tangent is 1.1106 is 48°.

EXAMPLE 3. A flagpole is side BC in $\triangle ABC$ pictured at the right. Point A is 42 feet from point C and $|\angle A|$ is 47°. How high is the flagpole?



The problem states the measure of an angle of a right triangle and the measure of the side adjacent to the given angle. You are to find the measure of the side opposite the given angle. Thus, you can use the tangent ratio.

$$\tan |\angle A| = \frac{|\overline{BC}|}{|\overline{AC}|}$$

$$\tan 47^\circ = \frac{|\overline{BC}|}{42}$$

From the table, $\tan 47^\circ$ is approximately 1.0724.

$$1.0724 \approx \frac{|\overline{BC}|}{42}$$

$$(1.0724)(42) \approx |\overline{BC}|$$

$$45.0408 \approx |\overline{BC}|$$

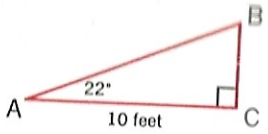
The height of the flagpole to the nearest foot is about 45 feet.

Now try these

Use the table of values on page 547 to find x.

1. $\tan 76^\circ \approx x$
2. $\tan 10^\circ \approx x$
3. $\tan x \approx 1.3270$
4. $\tan x \approx .5774$

5. Using $\triangle ABC$ at the right, find $\tan |\angle B|$ and $|\overline{BC}|$ to the nearest foot.



Answers: **1.** 4.0108 **2.** .1763 **3.** 53° **4.** 30° **5.** 2.4751; 4 feet

Checkpoint

1. What are similar triangles?
2. What is true about corresponding sides of similar triangles?
3. What is the definition of the tangent ratio?

Exercises

A — Use the table on page 547 to find the following.

1. tangent of 20° **2.** tan 22° **3.** tangent of 40°
4. tangent of 45° **5.** tan 55° **6.** tangent of 70°

— Find the tangent of each of the following angle measures.

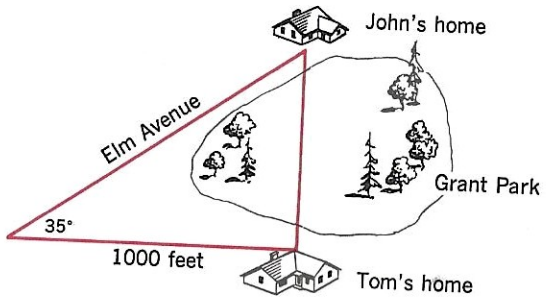
7. 5° **8.** 25° **9.** 37°
10. 80° **11.** 85° **12.** 89°

— Find x.

13. $\tan x \approx .5774$ **14.** $\tan x \approx 1.7321$ **15.** $\tan x \approx .1228$
16. $\tan x \approx .6009$ **17.** $\tan x \approx 1.6003$ **18.** $\tan x \approx 2.2460$

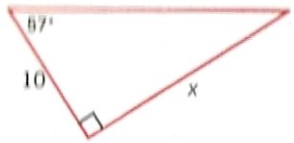
19. Refer to the table and complete the following sentence: As the measure of an angle increases from 0° to 89°, the tangent of the angle measure ? (increases or decreases) from .0000 to ?.

20. The triangle in the figure below is a right triangle. Use the tangent ratio to find the distance between Tom's home and John's home, directly through the park.

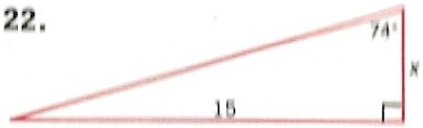


— Find the length, x, of the indicated side in each of the following figures.

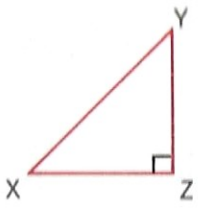
21.



22.



23. In right triangle XYZ, $|\angle X|$ is 45°. Since the sum of $|\angle X|$, $|\angle Y|$, and $|\angle Z|$ is 180°, $|\angle Y|$ is also 45°. If two angles of a triangle have the same measure, the sides opposite these angles have the same length and the triangle would be called an isosceles triangle.

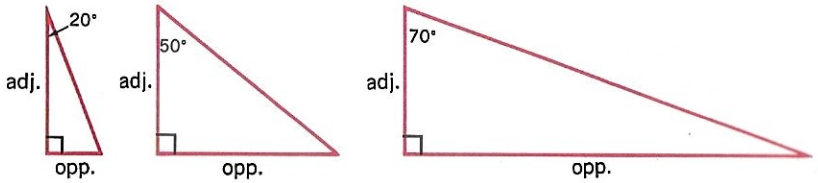


Triangle XYZ is isosceles and $|\overline{XZ}| = |\overline{YZ}|$ is true.

$$\tan |\angle X| = \frac{|\overline{YZ}|}{|\overline{XZ}|} = \text{?} \qquad \tan |\angle Y| = \frac{|\overline{XZ}|}{|\overline{YZ}|} = \text{?}$$

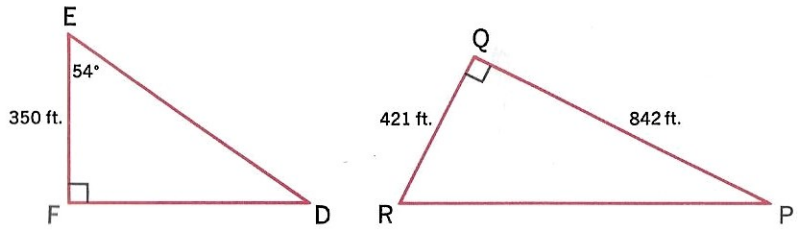
The tangent of 45° is equal to 1.

24. The tangent of 90° is not given in the table. Experiment with several triangles like those below to see what happens to the lengths of the opposite (opp.) and adjacent (adj.) sides as the angle gets larger.



Can you explain why the tangent of 90° is not given?

25. To the nearest foot, how long is $\overline{DF}$, in $\triangle DEF$ below?



B **26.** In $\triangle PQR$ above, what is $|\angle P|$, to the nearest degree?

27. Use Figure 1 below to find the length of $\overline{BC}$ to the nearest foot.

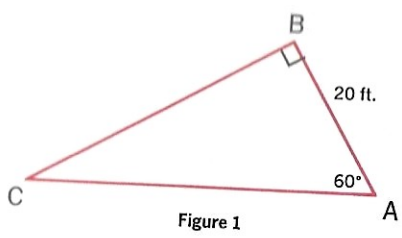


Figure 1

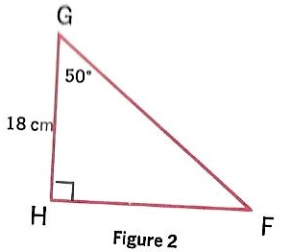
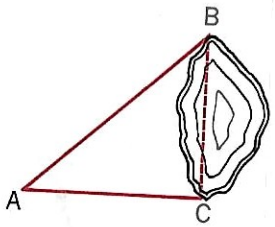


Figure 2

28. To the nearest centimeter, find the length of $\overline{HF}$ in Figure 2.

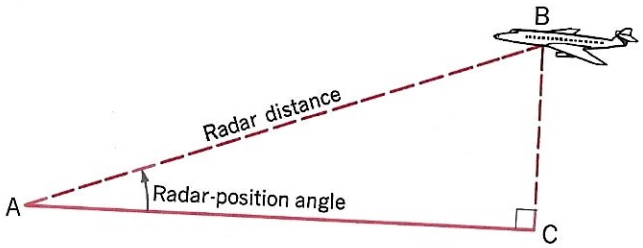
29. In the figure at the right, angle C is the right angle of $\triangle ABC$. What measurements could you make to find $|\overline{BC}|$, the distance across the pool?

30. Using the figure for Exercise 29, find $|\overline{BC}|$ if $|\angle A|$ is 42° and $|\overline{AC}|$ is 240 feet.



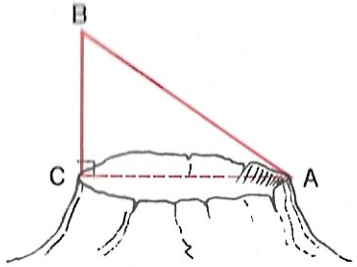
10–9 The Sine and Cosine Ratios

Suppose you wish to know the height of a plane, as illustrated in the figure below.



The radar reading gives you the distance from the radar tower to the plane, $|\overline{AB}|$, and the measure of angle A. Thus, you know the measure of an angle of a right triangle and the measure of the hypotenuse, and you are looking for the measure of the side opposite the known angle. This is not the information you would need to use the tangent ratio.

Or suppose your problem is to find the distance across the widest part of the mouth of a volcano. By considering $\triangle ABC$, you can find $|\angle A|$ and it is 35°, and you know that the measure of $\overline{AB}$, the hypotenuse, is 650 feet. The length you wish to find is that of the side adjacent to the known angle. This, too, is not the information you would need to use the tangent ratio.

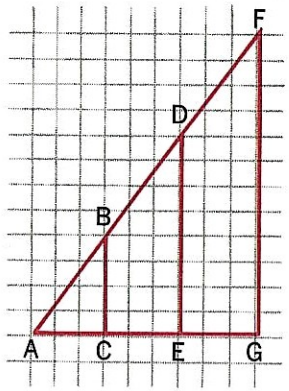


There are two other ratios you can use with the right triangle.

$$\frac{|\text{side opposite } \angle A|}{|\text{hypotenuse}|}$$

$$\frac{|\text{side adjacent to } \angle A|}{|\text{hypotenuse}|}$$

Apply these two ratios to the three right triangles shown at the right. You can find the lengths of the legs of the right triangles by counting the units. Then use the Pythagorean theorem to find the length of the hypotenuse.



Complete the following chart by replacing ? with your results.

	$\triangle ABC$	$\triangle ADE$	$\triangle AFG$
$\lvert$ side opposite $\angle A$ $\rvert$	$\lvert \overline{BC} \rvert = 4$	$\lvert \overline{DE} \rvert =$?	$\lvert \overline{FG} \rvert = 12$
$\lvert$ side adjacent to $\angle A$ $\rvert$	$\lvert \overline{AC} \rvert =$?	$\lvert \overline{AE} \rvert = 6$	$\lvert \overline{AG} \rvert =$?
$\lvert$ hypotenuse $\rvert$	$\lvert \overline{AB} \rvert = 5$	$\lvert \overline{AD} \rvert =$?	$\lvert \overline{AF} \rvert = 15$
$\frac{\lvert \text{opposite side} \rvert}{\lvert \text{hypotenuse} \rvert}$	$\frac{\lvert \overline{BC} \rvert}{\lvert \overline{AB} \rvert} =$?	$\frac{\lvert \overline{DE} \rvert}{\lvert \overline{AD} \rvert} =$?	$\frac{\lvert \overline{FG} \rvert}{\lvert \overline{AF} \rvert} =$?
$\frac{\lvert \text{adjacent side} \rvert}{\lvert \text{hypotenuse} \rvert}$	$\frac{\lvert \overline{AC} \rvert}{\lvert \overline{AB} \rvert} =$?	$\frac{\lvert \overline{AE} \rvert}{\lvert \overline{AD} \rvert} =$?	$\frac{\lvert \overline{AG} \rvert}{\lvert \overline{AF} \rvert} =$?

The three triangles in the figure are similar and, therefore, the ratios of the measures of their corresponding sides should be equal. Do your results support this?

Then, for any given acute angle of a right triangle, the ratio $\frac{|\text{opposite side}|}{|\text{hypotenuse}|}$ is always the same and the ratio $\frac{|\text{adjacent side}|}{|\text{hypotenuse}|}$ is always the same.

Definition For any given acute angle of a right triangle, the ratio of the measure of the side opposite the angle to the measure of the hypotenuse is called the sine.

Definition For any given acute angle of a right triangle, the ratio of the measure of the side adjacent to the angle to the measure of the hypotenuse is called the cosine.

Checkpoint

1. What is meant by the sine ratio?
2. What is meant by the cosine ratio?

Exercises

A — Use the table of values on page 547 to find approximations for the sine and the cosine of the following angles.

1. 35° **2.** 42° **3.** 54°
4. 83° **5.** 72° **6.** 61°

— Find x.

7. $\sin x \approx .1392$ **8.** $\sin x \approx .3907$ **9.** $\sin x \approx .6561$
10. $\sin x \approx .7547$ **11.** $\sin x \approx .9135$ **12.** $\sin x \approx .9962$

13. Look at your results for Exercises 7–12. As the size of an acute angle increases, does the sine ratio increase or decrease?

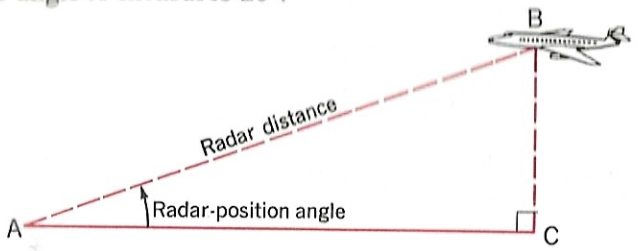
— Find x.

14. $\cos x \approx .9336$ **15.** $\cos x \approx .7660$ **16.** $\cos x \approx .5299$
17. $\cos x \approx .1736$ **18.** $\cos x \approx .0175$ **19.** $\cos x \approx .9877$

20. Look at the cosine values in the table. As the size of an acute angle increases, does the value of the cosine increase or decrease?

21. Now you can solve the problems mentioned at the beginning of the section.

a. The distance from the tower to the plane is 3000 feet, and angle A measures 20°.



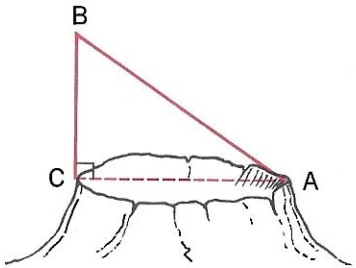
The height of the plane is the measure of the side opposite the 20° angle and you know the hypotenuse. Use the sine ratio.

$$\sin |\angle A| = \frac{|\overline{BC}|}{3000}$$

$$.3420 \approx \frac{|\overline{BC}|}{3000}$$

Solve for $|\overline{BC}|$.

b. Finding the distance across the mouth of the volcano is finding the measure of a side adjacent to a given acute angle of a right triangle, also given the measure of the hypotenuse. Use the cosine ratio.



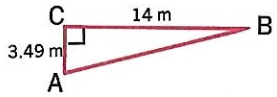
$$\cos 35° = \frac{|\overline{AC}|}{650}$$

$$.8192 \approx \frac{|\overline{AC}|}{650}$$

Solve for $|\overline{AC}|$.

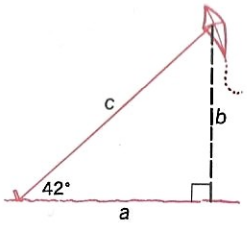
22. Use the triangle below to find the following values.

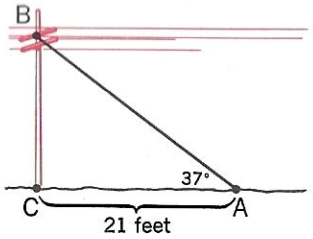
a. $\tan |\angle A| =$?
b. $\cos |\angle A| =$?
c. $\cos |\angle B| =$?
d. $\tan |\angle B| =$?
e. $\sin |\angle A| =$?
f. $\sin |\angle B| =$?



g. To the nearest degree, what is $|\angle A|$ and $|\angle B|$?

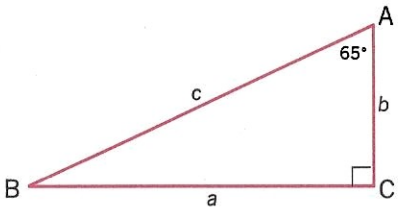
23. John's kite string is 196 feet long and makes an angle of 42° with the horizontal in the figure below. How high above the ground is the kite?

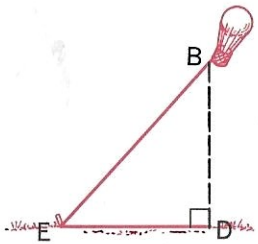




24. In the figure above, $\overline{AB}$ represents a wire supporting a telephone pole, $\overline{BC}$. Find $|\overline{AB}|$.

B **25.** In $\triangle ABC$ below, c is 75 feet and $|\angle A|$ is 65°. What is b?

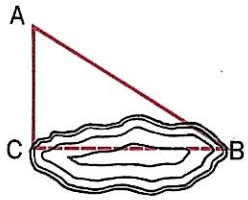




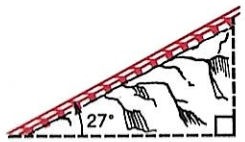
26. In the figure at the right above, the balloon at B is anchored to the ground at a point E by a wire. The wind blows the balloon so the wire makes an angle of 48° with the ground. If point D, directly under the balloon, is 260 feet from E, how long is the wire?

27. How far above the ground is the balloon in Exercise 26?

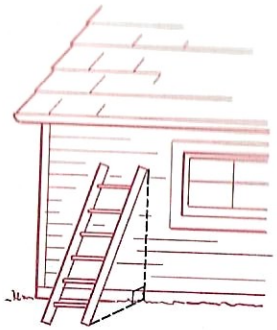
28. The scouts of the Wolf Patrol want to know the distance across a pond. They laid off $\overline{CA}$ at right angles to $\overline{BC}$. They extended $\overline{CA}$ until they came to a point A from which they could measure $\overline{AB}$. They measured $\overline{AB}$ and the angle A. If $|\overline{AB}|$ is 580 feet and $|\angle A|$ is 57°, find $\overline{BC}$.



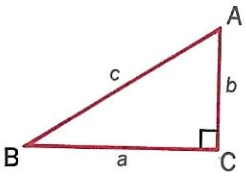
29. The railroad that runs to the summit of Pike's Peak makes, at the steepest place, an angle of 27° with the horizontal. How many feet would you rise in walking 100 feet up the railroad track?



30. The ladder in the figure is 20 feet long and reaches the wall at a point 18 feet from the ground. To the nearest degree, what is the measure of the angle that the ladder makes with the house?



Exercises 31–36 refer to *right triangles*. Make a drawing for each: a is the side opposite $\angle A$, b is the side opposite $\angle B$, and c is the side opposite $\angle C$ (which is a right angle).



31. Find c when a is 26 inches and $|\angle A|$ is 54°.

32. Find c when a is 8.4 centimeters and $|\angle A|$ is 62°.

33. Find a when c is 65 yards and $|\angle A|$ is 45°.

34. Find b when c is 100 feet and $|\angle A|$ is 35°.

35. Find b when c is 42 miles and $|\angle B|$ is 30°.

36. Find a when c is 78 meters and $|\angle A|$ is 54°.

37. If $|\overline{XY}|$ is 110 feet and $|\angle YXZ|$ is 27°, what is d in Figure 1?

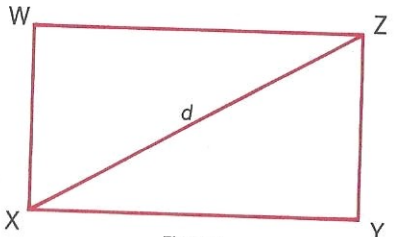


Figure 1

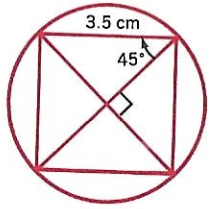


Figure 2

C **38.** The diagonals of a square are perpendicular to each other, as shown in Figure 2 above. If a circle is circumscribed about the square, the diagonals are also diameters of the circle. Find, by means of the cosine ratio, the radius of a circle circumscribed about a square whose side is 3.5 centimeters.

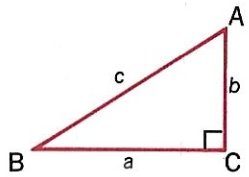
39. Use the Pythagorean theorem to check your answer to Exercise 38.

10–10 Solving Problems Using the Three Trigonometric Ratios

The following problems will give you practice in using all three trigonometric ratios — sine, cosine, and tangent. In solving these problems, you should first make a drawing, if no illustration is given, to show the parts that you know and the parts you wish to find. Then decide which ratio you should use to find the unknown part.

Exercises

A **1.** Refer to the figure given at the right and complete: Given b and $|\angle A|$, you can find a by using the ? ratio. Given b and $|\angle A|$, you can find the measure of the hypotenuse, c, by using the ? ratio. Given a and $|\angle A|$, you can find b by using the ? ratio, or you can find the measure of the hypotenuse, c, by using the ? ratio. Given $|\angle B|$ and a, you can find b by using the ? ratio, or you can find the measure of the hypotenuse, c, by using the ? ratio.

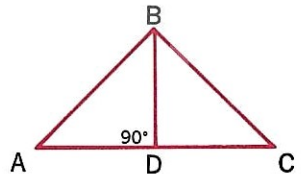


2. To find the distance across a pond between two points, M and N, Jack measured off 620 feet on a line MR perpendicular to segment MN. He found $|\angle MRN|$ to be 35°. Find $|\overline{MN}|$.

3. Some girl scouts measured the height of a mound. They stretched a string from a point, A, at the bottom of the mound to the top, T, finding $|\overline{AT}|$ to be 72.5 feet. The measure of the angle of elevation of the top from point A is 35°. How high is the mound?

4. If $|\overline{AB}|$ and $|\overline{BC}|$ in the figure at the right are each 30 feet and $|\overline{BD}|$ is 18 feet, what is the measure of angle A?

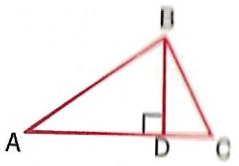
5. You are given rectangle $ABCD$ with diagonal $\overline{AC}$ drawn. The angle between $\overline{AD}$ and $\overline{AC}$ measures 31° and $|\overline{AD}|$ is 16.3 inches. How long is $\overline{AC}$?



B **6.** Find the angle of elevation of the sun to the nearest degree when a church steeple 200 feet high casts a shadow 80 feet long.

7. From the top of a cliff 3500 feet above a lake, the angle of depression of the nearest shore is 18°. Find the distance from the top of the cliff to the edge of the lake.

8. In the figure at the right, $|\overline{AB}|$ is 215 feet, $|\angle A|$ is 35°, $|\angle C|$ is 65°, and the angles at D are right angles. How long are $\overline{BD}$ and $\overline{BC}$?



10–11 Interpolation in a Trigonometric Table

The measure of an acute angle in a right triangle is not always a whole number of degrees. Values less than one degree are measured in minutes and seconds. Sixty minutes make one degree and sixty seconds make one minute. The expression 29° 34′ 11″ is read "twenty-nine degrees, thirty-four minutes, and eleven seconds." In such cases, it is not possible to read the value of the tangent, sine, or cosine of an angle directly from the table of values.

Similarly, if the value of a trigonometric ratio for an angle is not given in the table, it is not possible to read the measure of the angle directly from the table.

To find in-between values, a process called **interpolation** is used.

EXAMPLE 1. Find $\tan 22\frac{1}{2}°$.

Since $22\frac{1}{2}°$, or 22° 30′, is halfway between tan 22° and tan 23°, assume that tan 22° 30′ is halfway between tan 22° and tan 23°. With this assumption, you can set up a proportion using the differences between the values. The result will give you values that are satisfactory for most practical purposes.

① Write the values you are concerned with in table form.

$$\begin{aligned} \tan 22° &\approx .4040 \\ \tan 22°\ 30' &\approx \text{?} \\ \tan 23° &\approx .4245 \end{aligned}$$

② Consider 22° and 23° as 22° 00′ and 23° 00′, respectively. Find the differences between the angle measures and the tangent values. The differences are shown by the brackets.

$$60'\left[\begin{array}{l} 30'\left[\begin{array}{l} \tan 22°\ 00' \approx .4040 \\ \tan 22°\ 30' \approx \text{?} \\ \tan 23°\ 00' \approx .4245 \end{array}\right] x \end{array}\right] .0205$$

(Continued on next page.)

③ Then set up a proportion.

$$\frac{30}{60} \approx \frac{x}{.0205}$$
$$\frac{1}{2} \approx \frac{x}{.0205}$$
$$2x \approx .0205$$
$$x \approx .01025$$

Thus, .01025 is the difference between tan 22° and tan $22\frac{1}{2}°$.

④ Add the difference to the value of tan 22° to get the value of tan $22\frac{1}{2}°$.

$$\tan 22\tfrac{1}{2}° \approx .4040 + .01025$$
$$\approx .41425 \approx .4142$$

Note that here you added the difference because the tangent gets larger as the angle measure gets larger. But if this were the cosine, you would have to subtract the difference. Why?

EXAMPLE 2. What is the angle whose cosine is .8124?

① Look in the table to find two values between which .8124 falls and put your information in table form.

$$\cos 35° \approx .8192$$
$$\cos \underline{?} \approx .8124$$
$$\cos 36° \approx .8090$$

② Find the differences.

$$60' \left[x \left[\begin{matrix} \cos 35°\ 00' \approx .8192 \\ \cos \ \underline{?} \ \approx .8124 \end{matrix} \right] .0068 \right] .0102$$
$$\cos 36°\ 00' \approx .8090$$

③ Set up a proportion.

$$\frac{x}{60} \approx \frac{.0068}{.0102}$$
$$\frac{x}{60} \approx \frac{2}{3}$$
$$3x \approx 120$$
$$x \approx 40$$

④ The angle whose cosine is .8124 is approximately 35° 40′.

Always round your answers to the nearest whole number of minutes or, in the case of the decimal value, to a 4-place decimal.

Now try these

— Between what two values in the table will the following sine values fall?

1. .6049 **2.** .9175 **3.** .3201

4. Repeat Exercises 1–3, considering the values as cosine values.

— Between what two values in the table will the following tangent values fall?

5. 2.7841 **6.** .9056 **7.** .2511

8. Find the angle whose sine is .4722.

Answers: **1.** .6018 and .6157 **2.** .9135 and .9205 **3.** .3090 and .3256 **4.** .6157 and .6018; .9205 and .9135; .3256 and .3090 **5.** 2.7475 and 2.9042 **6.** .9004 and .9325 **7.** .2493 and .2679 **8.** 28° 11′

Exercises

A — By interpolation, find the value of each ratio.

1. tan $39\frac{1}{2}°$ **2.** cos $54\frac{1}{2}°$ **3.** sin $26\frac{1}{4}°$
4. tan 38.7° **5.** sin 73.4° **6.** cos 72.6°

— By interpolation, find x.

7. $\cos x = .9763$ **8.** $\cos x = .6364$ **9.** $\cos x = .8039$
10. $\sin x = .2840$ **11.** $\sin x = .8351$ **12.** $\sin x = .9542$
13. $\tan x = 2.2208$ **14.** $\tan x = .2357$ **15.** $\tan x = 1.0488$

16. How tall is a tree that casts a shadow of 60 feet when the angle of elevation of the sun is $35\frac{1}{2}°$?

17. Figure 1 below shows a triangle inscribed in a circle; $|\angle OAB|$ is 30°, $\overline{OB}$ is perpendicular to $\overline{AC}$, and $|\overline{AB}| = \frac{1}{2}|\overline{AC}|$. How long is side AC if the radius of the circle is 6.3 inches?

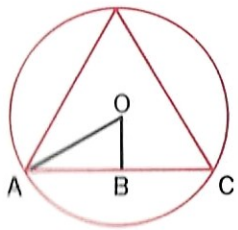


Figure 1

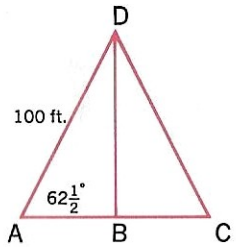


Figure 2

18. In Figure 2 above, how long is $\overline{AC}$ if $\overline{DB}$ is perpendicular to $\overline{AC}$ and $|\overline{AB}| = |\overline{BC}|$?

CHAPTER REVIEW

1. Know the meaning of and be able to use each of the following words or phrases. The number shown after each word or phrase indicates where it is introduced, in case you need to review.

trigonometry (*434*)
direct measurement (*435*)
indirect measurement (*435*)
angle (*436*)
sides (*436*)
vertex (*436*)
right angle (*436*)
acute angle (*437*)
obtuse angle (*437*)
right triangle (*437*)
hypotenuse (*437*)
ratio (*439*)
similar triangles (*440*)
tangent (*440*)
sine (*446*)
cosine (*446*)
interpolation (*451*)

2. By what do you multiply both sides of the following equation to obtain an equation without rational expressions?

$$\frac{x}{3}+\frac{x}{4}=\frac{13}{12}$$

3. What is the lowest common denominator for $\frac{8}{x-2}=\frac{7}{x-7}$? Solve the equation.

Solve the following equations.

4. $\frac{7x-2}{4x}-\frac{2x+3}{3x}=\frac{5}{12}$

5. $\frac{x-2}{x+3}=\frac{x+5}{x-7}$

6. The numerator of a rational number is 2 smaller than the denominator. If the numerator is increased by 9 and the denominator is multiplied by 3, the rational number equals $\frac{2}{3}$. What is the number?

7. The Mammouth Dairy has an old machine that fills cartons with milk at the rate of 1000 in 15 minutes. Their newer machine fills 1000 cartons in 9 minutes. How long will it take both machines working together to fill 1000 cartons with milk?

8. Mr. Deluca drove 400 miles in heavy traffic going on a vacation trip. He returned home in much lighter traffic and was able to drive at a rate that was a fourth faster. The return trip took him two hours less time. What was his rate of speed going on the trip?

9. What operation do you perform to change each equation into the next equivalent equation?

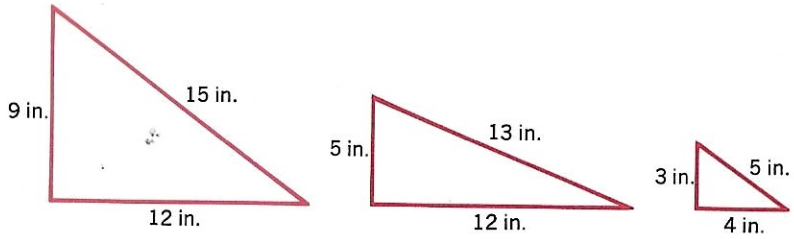
$$x+\frac{y}{5}=y \rightarrow 5x+y=5y \rightarrow -4y=-5x \rightarrow y=\frac{5x}{4}$$

10. The formula for the area of a trapezoid is $A=\frac{h(B+b)}{2}$. Solve this formula for B.

11. Before cutting down a large tree in his yard, Mr. Jacobs measured the circumference of its trunk and computed its diameter by using the formula $C=\pi d$. After cutting down the tree, he measured the diameter as 27 inches. Which was an indirect measurement? Should the two measures have been the same?

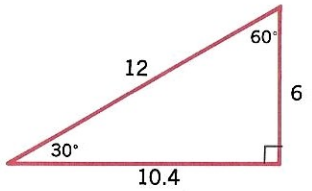
12. What is the name of an angle whose measure is 90°? whose measure is greater than 90°? less than 90°?

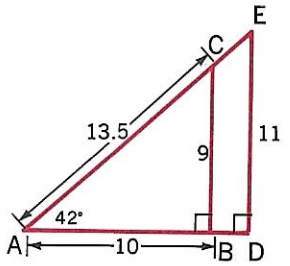
13. Which two of the following triangles are similar?



For each of the three pairs of corresponding sides of the similar triangles, write the ratio of the lengths. Are the three ratios equal?

14. For the triangle at the left below, write the ratio that represents tan 30°; sin 60°; cos 30°.





15. Look at triangle ABC above. What is $\tan |\angle A|$? Does the ratio $\frac{9}{10}$ approximately represent tan 42°? Does the ratio $\frac{11}{|\overline{AD}|}$ represent tan 42°? What is $|\overline{AD}|$? What is $|\overline{BD}|$?

Draw $\triangle ABC$, $|\angle B| = 90°$. Then answer Exercises 16–19 using $\triangle ABC$.

16. If you are given $|\angle A|$ and $|\overline{AB}|$, would you use the tangent, sine, or cosine ratio to find $|\overline{BC}|$?

17. Given $|\angle A|$ and $|\overline{BC}|$, what ratio would you use to find $|\overline{AC}|$?

18. Given $|\angle A|$ and $|\overline{AC}|$, what ratio would you use to find $|\overline{BC}|$? $|\overline{AB}|$?

19. Given $|\angle C|$ and $|\overline{BC}|$, what ratio would you use to find $|\overline{AC}|$?

20. Using the table on page 547, interpolate to find x to the nearest 10 minutes if $\tan x = 2.8355$.

21. Interpolate to find $\sin 54.8°$ to the nearest ten-thousandth.

CHAPTER TEST

1. The formula $V = \frac{1}{3}\pi r^2 h$ is used to find the volume of a cone. Express this formula in terms of h.

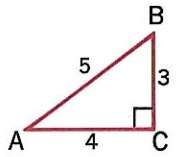
2. To determine the total surface area of a cylinder, you can use the formula $T = 2\pi r(r + h)$. Express this formula in terms of h.

Solve the following equations for x.

3. $\frac{2}{x} = \frac{6}{x+5}$ **4.** $\frac{4-x}{4} = \frac{6}{2-x}$ **5.** $\frac{x-2}{x+4} = \frac{x-1}{x-3}$

6. One number is 6 less than the other. If 5 is subtracted from both numbers, the quotient of the larger divided by the smaller is $\frac{4}{3}$. What are the numbers?

7. Using the triangle at the right, write each of these ratios.



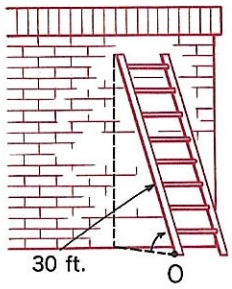
a. $\sin |\angle A| = \underline{?}$ **b.** $\tan |\angle A| = \underline{?}$

c. $\cos |\angle A| = \underline{?}$ **d.** $\tan |\angle B| = \underline{?}$

e. $\sin |\angle B| = \underline{?}$ **f.** $\cos |\angle B| = \underline{?}$

8. One angle of a right triangle measures 70° and the side adjacent to it measures 19 feet. To the nearest tenth of a foot, what is the measure of the side opposite the 70° angle?

9. If you want the top of the 30-foot ladder shown to be 25 feet from the ground, to the nearest degree, what must the measure of angle O be? If you want the bottom to be 20 feet from the building, what must the measure of angle O be?

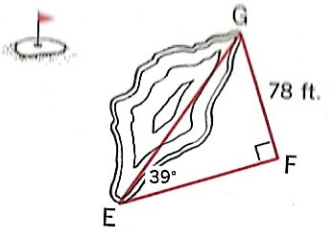


10. Jack can mow his lawn in 6 hours. His younger brother Frank can mow it in 10 hours. How long will it take them to mow the lawn if they work together?

11. Find $\sin 56° 50'$.

12. Find $\cos 28° 30'$.

13. Shady Golf Course has a very large water trap. Using the figure at the right, find the length, $|\overline{GE}|$, of the water trap to the nearest foot.



ENRICHMENT

Convergents of Continued Fractions

As you saw in Chapter 9, a continued fraction has the form

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\quad + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}}$$

For $1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}}$,

$\frac{p}{q}$ is $\frac{225}{157} \approx 1.433121$, which you found by adding from the bottom up. Suppose you add from the top down. What values do you get? Call the values of these partial quotients c_n.

$c_1 = 1$ $\quad c_2 = 1 + \frac{1}{2}$, or $\frac{3}{2}$ $\quad c_3 = 1 + \cfrac{1}{2 + \frac{1}{3}} = 1 + \frac{1}{\frac{7}{3}}$, or $\frac{10}{7}$

$c_4 = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{4}}} = 1 + \cfrac{1}{2 + \frac{1}{\frac{13}{4}}} = 1 + \frac{1}{2\frac{4}{13}} = 1 + \frac{1}{\frac{30}{13}}$, or $\frac{43}{30}$

$c_5 = \frac{225}{157}$

If you expressed each value as a decimal, you would see that the values are getting closer and closer to 1.433121. The values are *converging* to c_5. For this reason, the partial quotients are called the **convergents** of the continued fractions. The convergents have some interesting properties and they can be used to solve equations.

The general form of the convergents is $c_1 = \frac{p_1}{q_1} = a_1$; $c_2 = \frac{p_2}{q_2} = a_1 + \frac{1}{a_2}$; $c_3 = \frac{p_3}{q_3} = a_1 + \frac{1}{a_2 + \frac{1}{a_3}}$; $c_4 = \frac{p_4}{q_4} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4}}}$; $c_5 = \frac{p_5}{q_5} = \frac{p}{q}$.

The following is a chart of the convergents of $\frac{225}{157}$.

n	1	2	3	4	5
a_n	1	2	3	4	5
p_n	1	3	10	43	225
q_n	1	2	7	30	157

An interesting pattern results if you cross-multiply, following the arrows, and subtract the products.

$$q_1p_2 - q_2p_1 = (1 \cdot 3) - (2 \cdot 1) = 1$$
$$q_2p_3 - q_3p_2 = (2 \cdot 10) - (7 \cdot 3) = -1$$
$$q_3p_4 - q_4p_3 = (7 \cdot 43) - (30 \cdot 10) = 1$$
$$q_4p_5 - q_5p_4 = (30 \cdot 225) - (157 \cdot 43) = -1$$

The pattern of 1 and -1 is true for convergents in general, although this will not be proved here. The pattern can be represented by the formula $q_np_{n+1} - q_{n+1}p_n = (-1)^{n+1}$ and is a help in solving equations.

Suppose you are given $5m + 7n = 3$, an equation of the form $am + bn = c$, where a, b, and c are integers, and you want to find integers for m and n that make the equation true. In a simple case like this, you can determine by trial and error that $m = 2$ and $n = -1$ satisfy the equation, and so do $m = 9$ and $n = -6$.

But it might take quite a few trials and errors to solve $31x + 11y = 2$. A more definite method of solution is needed. You can use continued fractions to develop a method for solving such linear equations.

Use the coefficients of the variables to form a fraction greater than 1; in this case it is $\frac{31}{11}$. Then expand this fraction into a continued fraction, find the convergents, and make a chart as at the right.

n	1	2	3	4
a_n	2	1	4	2
p_n	2	3	14	31
q_n	1	1	5	11

Cross-multiply and subtract using the equation $q_np_{n+1} - q_{n+1}p_n = (-1)^{n+1}$.

$$(1 \cdot 3) - (1 \cdot 2) = 1$$
$$(1 \cdot 14) - (5 \cdot 3) = -1$$
$$(5 \cdot 31) - (11 \cdot 14) = 1$$

Notice that if the terms of the last equation on the previous page are slightly rearranged, you have

$$31(5) - 11(14) = 1,$$

which corresponds very closely to the original equation. Substituting a for 31 and b for 11, you may write $5a - 14b = 1$. Then multiplying by 2 and changing the sign of the b term gives $10a + (-28)b = 2$, which is the same as $10(31) + -28(14) = 2$ and the problem is solved: $m = 10$ and $n = -28$.

Exercises

1. Calculate the convergents for the following continued fraction. Fill in a chart and check $q_np_{n+1} - q_{n+1}p_n = (-1)^{n+1}$.

$$2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{1}{5}}}}}$$

2. Use continued fractions to find integral solutions for the following equations. Follow the example in the discussion.

a. $17m - 6n = 1$ **b.** $54m + 13n = 2$

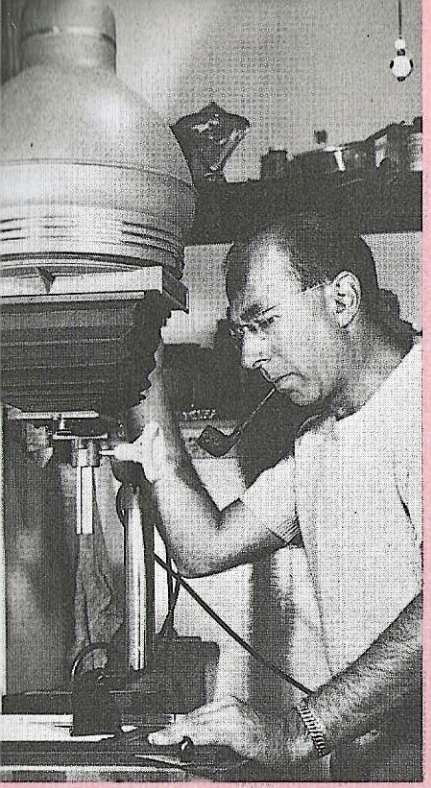
c. $31m - 15n = 1$ **d.** $74m - 253n = 1$

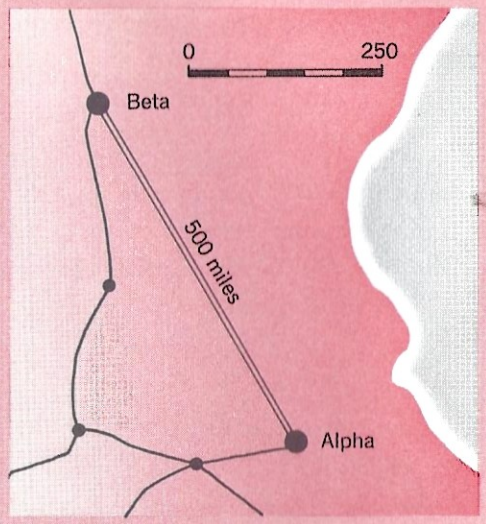
THE GROWTH OF MATHEMATICS

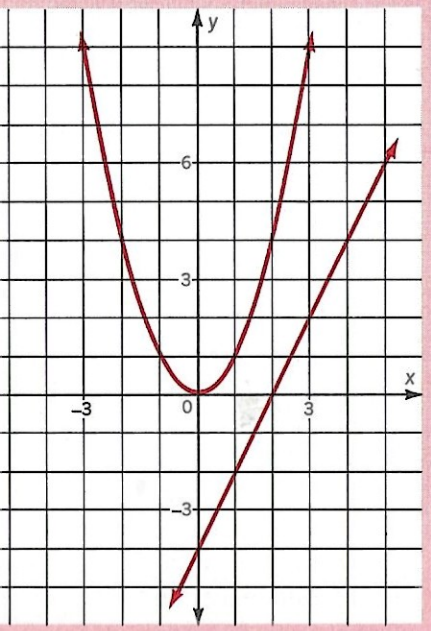
New Ideas in Trigonometry

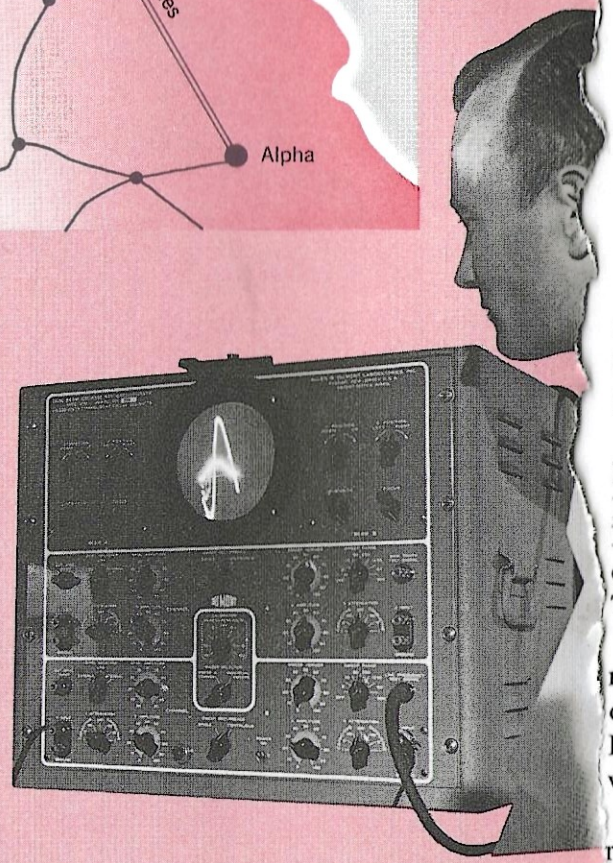
The first trigonometric tables were done before Euclid's time, and trigonometry was actively studied and improved by various mathematicians in Alexandria and in India. The chief purpose of trigonometry was to aid in astronomical calculations.

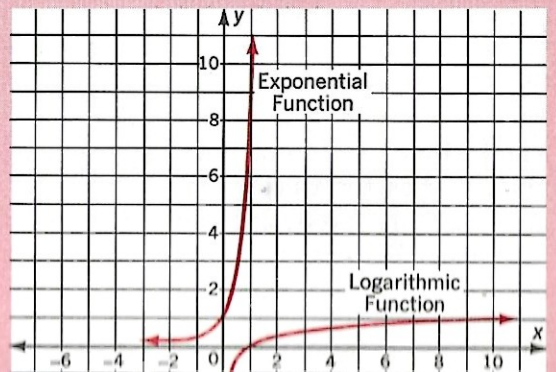
Johann Müller (1436–1476), a German astronomer, reintroduced the study of trigonometry in European mathematics. Müller is usually known as Regiomontanus, the Latin name of the city of Königsberg, where Müller was born. One of the contributions of Müller to trigonometry was the calculation of tables for very small differences in measure, thus producing more precise values of the trigonometric functions. He was also actively involved in collecting and translating Greek mathematical works that had been preserved in manuscript.











CHAPTER 11

RELATIONS AND FUNCTIONS

Relations and functions are among the most interesting and most useful parts of mathematics. In this chapter you will learn some of the general properties of relations and functions as well as the specific properties of a few special relations and functions.

- In the upper left region on the opposite page, a man is enlarging a photograph. Think of a ray of light from the bulb in the enlarger passing through the negative. Does the ray suggest a point on the enlargement that corresponds to a point on the negative? Is every point on the negative associated with a point on the enlargement by a ray? This illustrates the idea of a mapping.
- Look at the map in the upper right region. You know that the relation between distance, rate, and time is $d = rt$. At an average rate of 100 mph, how many hours will it take for the trip from Alpha to Beta? how many hours at 10 mph? This example illustrates inverse variation.
- Two functions that you have studied are graphed in the lower left region. How can you show that each is a function? What is the name of each?
- The graphs shown in the lower right region are of functions that are probably new to you. Trace each with your finger. How do the graphs differ from those at the left? How is the graph of the exponential function similar to the graph of the logarithmic function? How do they differ?
- The oscilloscope in the center of the page displays a curve that is the graph of the sine function. This curve often appears in nature as well as in science and mathematics.

11–1 Mappings

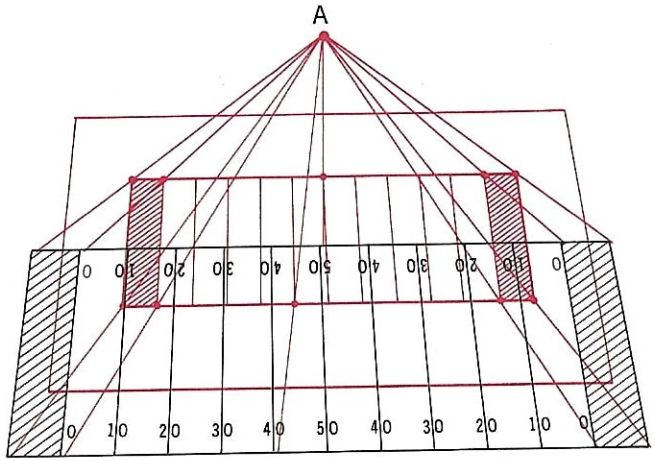
You are familiar with maps of various kinds, such as road maps, political maps, and weather maps. The primary purpose of a map is to represent on a smaller (or larger) scale some particular characteristic of the object being mapped — direction and type of highway for the road map; relative position and size of countries for the political map; and temperature, humidity, and air pressure for the weather points in the drawing. The way the map is made can be thought of as the **rule** for making the association between points of the real object and the points on the map.

EXAMPLE 1. Give a method for mapping a football field.

Rule

① Choose a point above the center of the field. Label it A.

② Choose a plane parallel to the field between A and the field. The position of the plane will determine the size of the map.



③ Draw the line segments from A to points on the field.

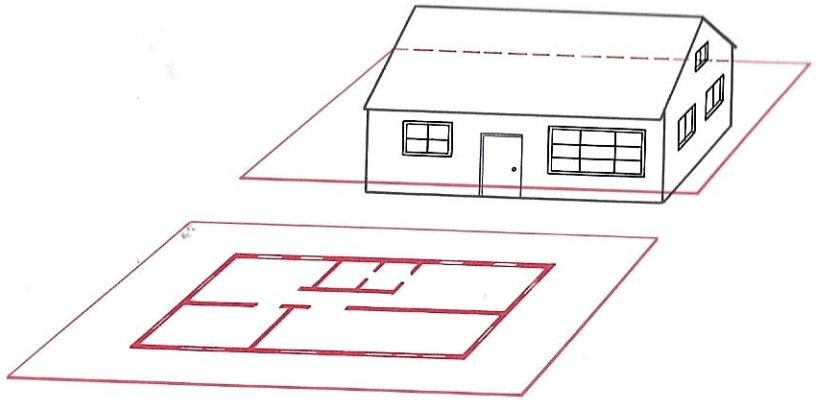
④ The intersection of the plane and the set of these segments is the map.

Each point of the football field is mapped to exactly one point in the drawing according to this plan. Note that the map will be smaller than the football field. How would you change the plan to get an enlargement of the football field?

EXAMPLE 2. Give a plan for mapping the rooms in a one-story house.

Rule

① Choose a plane passing through the house parallel to the floor at a height such that it passes through each window and each door.



② The intersection of the plane and the house is the map.

Many points of the house are mapped into a single point on the map. For example, all of the points in a segment perpendicular to the floor become one point on the map. Also, the map is the same size as the floor of the house. You could reduce the size by repeating the method described in Example 1.

The two examples illustrate the mathematical idea of a **mapping.**

Definition A mapping is a rule that associates the elements of one set, called the domain, with elements from a second set. The second set associated with elements in the domain is called the range or the image.

Notice that there are three components in the definition: two sets and one rule. In Example 1, the first set, or domain, is the set of points on the football field. The other set, or range, is the set of points on the map. The rule is the way the mapping was done.

In Example 1, each point in each set was mapped into one and only one point in the other set. Such a mapping is called a one-to-one mapping.

Definition A mapping is one-to-one whenever each element in the range is the image of one and only one element in the domain and each element in the domain maps into one and only one image element in the range.

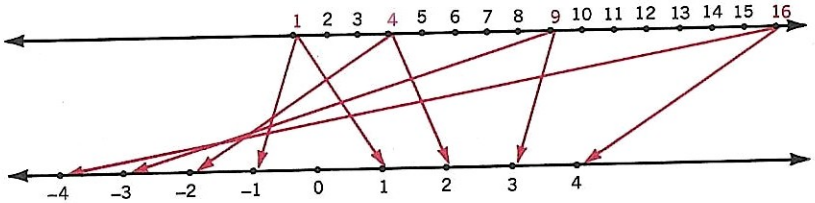
Use the definition to explain why Example 1 is a one-to-one mapping. Is Example 2 a one-to-one mapping of the house into the map?

Definition A mapping is many-to-one when at least two elements in the domain are matched with one element in the range.

Explain why Example 2 is a many-to-one mapping. Example 3 illustrates a one-to-many mapping.

EXAMPLE 3. Illustrate the mapping for which the set of positive integers is mapped into the set of integers with each element in the first set matched with its square root from the second set.

The domain is the set of perfect-square integers, the range is the set of integers except 0, and the rule is to associate each element in the domain with its square root.



In this case each perfect square is mapped to two integers. Notice that only perfect squares are mapped. Explain why 0 is not in the image set.

Definition A mapping is one-to-many when at least one element in the domain is matched with two or more elements in the range.

Checkpoint

1. What three components are needed to define a mapping?
2. What is the meaning of the word *many* in the expression *one-to-many*?
3. What is the difference between one-to-many and many-to-one?
4. Describe the conditions for a one-to-one mapping.

Exercises

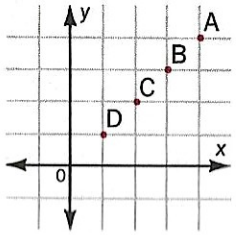
A Give the range for each mapping. State whether each mapping is one-to-one, one-to-many, or many-to-one. Draw diagrams for each exercise similar to that in Example 3 on page 464.

	Domain	Second set	Rule
1.	{0, 2, 4, 6, 8, 10}	Positive integers	Add 1.
2.	{1, 2, 3, 4, 5}	Positive integers	Multiply by 5.
3.	{25, 20, 15, 10}	Positive integers	Divide by 5.
4.	{−7, 3, 5, 7, −3, 0}	Integers	Take absolute value.
5.	{2, 7, 9, 8}	Real numbers	Take principal square root.

6. Suppose the domain of a mapping is the set of children in the United States, the range is the set of men in the United States, and the rule is *Fathers are matched with children.* How would you describe the mapping: as one-to-one, as many-to-one, or as one-to-many?
7. Let the set of cars on the New York State Thruway form the domain and the set of tires form the range. *Cars are matched with the tires on them.* How would you describe the mapping?
8. Is there a one-to-one mapping of the real numbers with the points on a line? Explain.

B 9. A graph is the image that results when pairs of numbers are mapped with points in the plane. Look at the drawing.

Domain: {(1, 1), (2, 2), (3, 3), (4, 4)}
Image: {A, B, C, D}



On to what point is each pair mapped?

10. Is there a one-to-one mapping of the even numbers with the set of positive integers? Explain.

11. Is there a one-to-one mapping of the rational numbers with the set of points on a line? Explain.

12. Is there a one-to-one mapping of the set of ordered pairs of real numbers with the points on a plane?

— Sketch a graph of the mapping defined by the equation or inequality when the set of real numbers is mapped with the set of real numbers according to the following rules. In each case, give the domain and range.

13. $y = |x|$ **14.** $y = |x + 2|$

15. $|y| = x$ **16.** $|y + 2| = x$

17. $y = x^2$ **18.** $x = y^2$

19. $y > x^2$ **20.** $x + y \leq 10$

21. Which mappings in Exercises 13–20 are one-to-one? one-to-many? many-to-one?

C — Think of the possible ordered pairs formed using the given domain and range. Then write the ordered pairs in the relation defined by the rule.

	Domain	Range	Rule
22.	$\{5, 10, 12\}$	$\{1, 11\}$	$>$
23.	$\{8, \sqrt{19}\}$	$\{2\sqrt{5}, \sqrt{64}, \sqrt{81}\}$	$<$
24.	$\{5\sqrt{3}, -2\sqrt{6}\}$	$\{\sqrt{24}, \sqrt{75}\}$	$\neq$

11–2 Ordered Pairs

A *relation* has been defined as a set of ordered pairs. A *function* has been defined as a relation with the first elements in the pairs all different. Shown below are some relations involving pairs of numbers. Why is the relation R_1 not a function? Why are R_2, R_3, and R_4 relations that are also functions?

a. $R_1 = \{(10, 9), (10, 8), (10, 7), (10, 6), (10, 5)\}$

b. $R_2 = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$

c. $R_3 = \{(-3, 3), (-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2), (3, 3)\}$

d. $R_4 = \{(-5, -3), (-3, -1), (-1, 1), (1, 3), (3, 5), (5, 7)\}$

Each relation in **a–d** can be shown by using an equation or inequality and set-builder notation. Which set at the right has each relation in **a–d** as a subset?

e. $\{(x, y) : y = x + 2\}$
f. $\{(x, y) : y = |x|\}$
g. $\{(x, y) : y = x^2\}$
h. $\{(x, y) : y < x\}$

The sets defined in **e–h** show that some relations can be defined by equations. In fact, the two functions you have studied most thoroughly, the *linear* and *quadratic*, are defined by using equations. A *linear* function is defined as the set of ordered pairs (x, y) such that $y = mx + b$, $m \neq 0$ and $x \in R$, and the *quadratic* function is defined as $y = ax^2 + bx + c$, $a \neq 0$ and $x \in R$.

It appears that relations and mappings have a lot in common. In fact, every relation can be thought of as a mapping, and vice versa. The *domain* of a relation is simply the set of first elements of the pairs. The *range*, or *image*, is the set of second elements. What are the domain and range in each of **a–d**?

The domain and range of a relation need not be sets of numbers.

i. $R_5 = \{$(New York, Albany), (Michigan, Lansing), (California, Sacramento), (Virginia, Richmond)$\}$

j. $R_6 = \{$(General Motors, Chevrolet), (Ford, Lincoln), (Chrysler, Plymouth), (American, Rambler)$\}$

What is the domain and range of each relation in **i** and **j**? Give a verbal rule describing the domain; the range. Is each a function?

Checkpoint

1. What is a relation? a function?

2. Can *some* relations be defined by equations or inequalities? Can *all* relations be defined by equations or inequalities?

3. What is the replacement set for the linear function? the quadratic function? How is each function defined?

Exercises

A — List the domain and range for each relation in Exercises 1–4.

1. $\{(1, 10), (1\frac{1}{4}, 10), (1\frac{1}{2}, 10), (1\frac{3}{4}, 10), (1\frac{7}{8}, 10)\}$

2. $\{(1, 1), (4, 2), (9, 3), (16, 4), (25, 5)\}$

3. $\{$(Indiana, Indianapolis), (Ohio, Columbus), (Alabama, Montgomery), (Pennsylvania, Harrisburg)$\}$

(feathers, bird), (hair, dog)}

5. Which relations in Exercises 1–4 are functions?

6. Give a mathematical sentence or a verbal description of the way the pairs were formed in Exercises 1–4.

7. The equation $y = 3x + 4$ defines a relation between x and y. Complete the table showing seven number pairs that satisfy $y = 3x + 4$. Then draw the graph of this portion of the relation on the x and y axes. The replacement set for x is $\{0, 1, 2, 3, 4, 7, 10\}$.

When x is	0	1	2	3	4	7	10
then y is	4	7	?	?	?	?	?

Is this relation a function?

8. The perimeter of a regular hexagon is related to the length of one side. The relation is defined by $p = 6s$. Complete the table below showing several pairs of numbers that satisfy $p = 6s$ when $s \in \{0, 1, 2, 3, 4, 5, 6\}$. Then graph the relation with s on the horizontal axis and p on the vertical axis.

When s is	0	1	2	3	4	5	6
then p is	0	6	?	?	?	?	?

The tables below define relations. Study each table and then write an equation that defines each relation. Which relations are functions?

9.

n	3	4	5	6	7	8
d	0	1	2	3	4	5

10.

w	1	2	3	4	5	6
c	5	10	15	20	25	30

11.

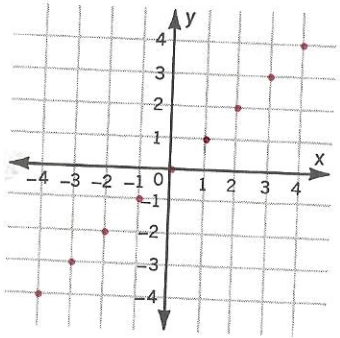
q	2	3	4	5	6	7
p	5	7	9	11	13	15

12.

x	0	1	2	3	4	5	6
y	−1	1	3	5	7	9	11

List the set of ordered pairs of numbers for each relation graphed below. Is the relation a function?

13.



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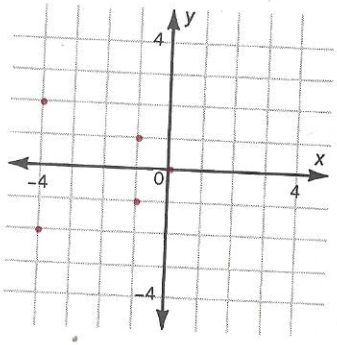
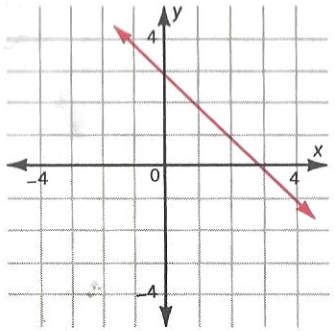


Sketch the graph on a plane of the relations defined by the following inequalities if the replacement set is the set of real numbers.

15. $y < -1$ 16. $x > 2$ 17. $-3 < |x| < 3$
18. $y > 1$ 19. $x < -2$ 20. $0 < |y| < 1$

Below are graphs of six relations. Which are functions?

21.



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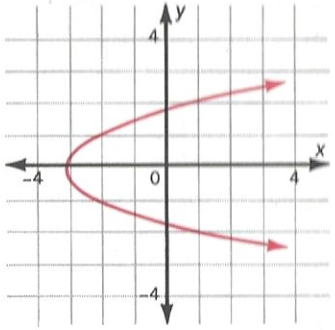

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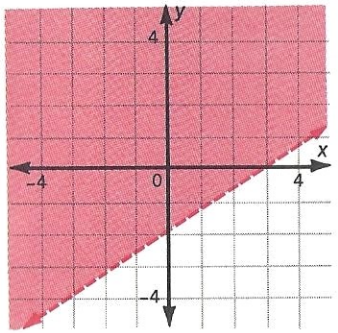

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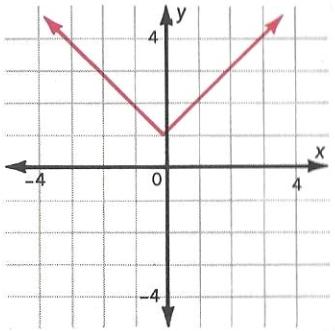

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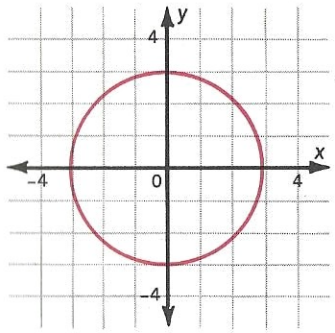
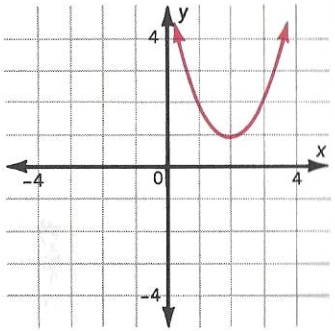


26.



Which of the following sentences define a function?

27. $y = 7x - 9$ 28. $x = y + 8$ 29. $y < x$
30. $0 \cdot y + x = 9$ 31. $y = x^2 + 7$ 32. $x = y^2$

B — Graph each of the following relations where the domain is $\{-4, -3, -1, 0, 1, 2, 3, 4\}$. Indicate which are functions.

33. $\{(x, y) : y = x\}$ **34.** $\{(x, y) : |x| = 3y\}$

35. $\{(x, y) : x + y \geq 1\}$ **36.** $\{(x, y) : x^2 = y\}$

37. In the Bunch family there are three young children, Willy, Sally, and Jim. Suppose the replacement set for the elements in the ordered pairs is the set of children. What ordered pairs would result using *is a brother of* to define the relation between the first and second elements? Is the relation a function?

— Using the same children as in Exercise 37 and Mr. Bunch, list the ordered pairs for each relation defined as follows.

38. is the father of **39.** is the sister of

40. is the son of **41.** is the daughter of

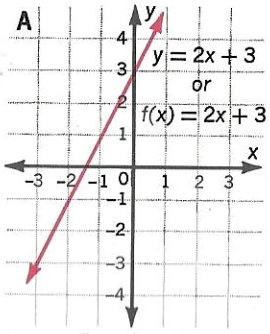
42. is the mother of

43. Which relations in Exercises 38–42 are functions?

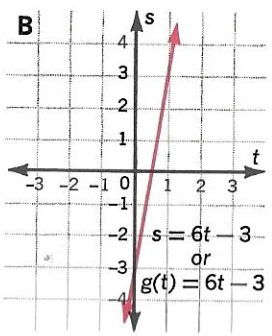
11–3 Graphs of Relations

You learned earlier that a relation defined by a set of ordered pairs can be graphed. When the relation is defined by an equation, the graph of the relation is found by plotting all the ordered pairs that satisfy the equation.

A relation may be symbolized with a letter such as R, f, g, h, or some other letter. You might use f to denote the relation defined by the equation $y = 2x + 3$, which is shown in graph A below. The letter g might denote the relation graphed in B.



Function: f



Function: g

When a relation is a function, there is a special way of showing range elements. For the function f the image of 2 is 7, since $2(2) + 3$ is equal to 7. This is shown by writing $f(2) = 7$, read "f at two equals seven." In general, $f(x) = y = 2x + 3$ describes the function, so $f(x)$ is another way to denote y. Notice that $f(x)$ does *not* mean "f times x" when f is the name of a function. In B, the expression $g(t)$ means the value of g from the range for the given value of t, so $g(t) = 6t - 3$ is another way to describe the function.

Notice that this notation is usually restricted to functions. If this notation is used for a relation that is not a function, say $u(v) = \pm v$, then $u(2)$ denotes the set $\{-2, 2\}$.

Can you identify the type of function graphed in A and B on the previous page? It is the linear function. Recall its definition and also that of the quadratic function.

A linear function is a function defined by the equation $y = mx + b$, where m, b, and x are real numbers, $m \neq 0$.

A quadratic function is a function defined by the equation $y = ax^2 + bx + c$, where a, b, c, and x are real numbers, $a \neq 0$.

It can be proved that every linear function has a graph that is a line; and every line in the coordinate plane not parallel to the y axis is the graph of a linear function. Similarly, the graph of every quadratic function is a parabola; and every parabola whose axis of symmetry is parallel to the y axis is the graph of a quadratic function.

The graph of any relation with numbers may be drawn by plotting the ordered pairs on the coordinate plane. Many numerical relations are neither linear nor quadratic, but are easily graphed.

EXAMPLE 1. Graph the function defined by the following.

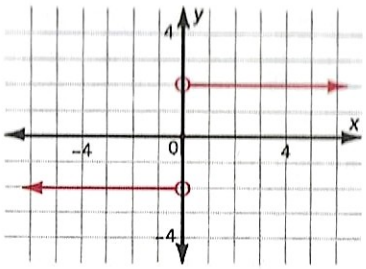
$$g(x) = \begin{cases} -2 \text{ when } x < 0 \\ 0 \text{ when } x = 0 \\ 2 \text{ when } x > 0 \end{cases}$$

By looking at the values x can have, you can see that the domain is the set of real numbers. The range, or image set, of the function is $\{-2, 0, 2\}$. Some typical ordered pairs are given in the table below.

x	−50	−2	−0.0001	0	0.0001	5	200
$g(x)$	−2	−2	−2	0	2	2	2

Explain why g is a function. Is it one-to-one? The graph is on the following page.

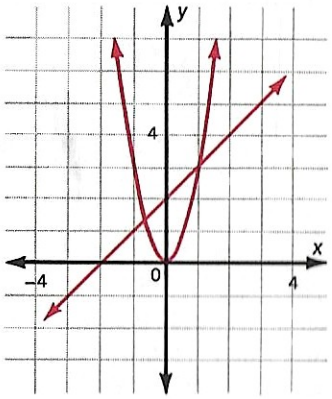
Here is the graph of g.



EXAMPLE 2. Graph the relation defined by

$$S = \{(x, y) : y = x + 2 \text{ or } y = 3x^2\}.$$

If a relation is defined by two or more equations joined by *or*, the graph is found by graphing each equation. The union of all these points is the graph of the relation. Using union symbolism, the relation is $S = \{(x, y) : y = x + 2\} \cup \{(x, y) : y = 3x^2\}$.



Is the graph in Example 2 the same as the graph of this relation?

$$\{(x, y) : y = x + 2 \text{ and } y = 3x^2\}$$

How do you graph sentences joined by *and*?

Now try these

If $f(x) = 2x + 6$, find

1. $f(-1)$. **2.** $f(\frac{1}{2})$. **3.** $f(p)$.

If $f(x) = x^2 - 2x + 6$, find

4. $f(0)$. **5.** $f(1)$. **6.** $f(-2)$.

If $g(x) = \frac{2x}{x+3}$, find

7. $g(0)$. **8.** $g(0.25)$. **9.** $g(-3)$.

10. Describe the graph of a linear function.

11. Describe the graph of a quadratic function.

Answers: **1.** 4 **2.** 7 **3.** $2p + 6$ **4.** 6 **5.** 5 **6.** 14 **7.** 0 **8.** $\frac{2}{13}$, or 0.153846 **9.** -3 is not an element of the domain, so $g(-3)$ is not defined.

Exercises

A

If g is the function defined by $y = -2x + 3$, find

1. $g(0)$. **2.** $g(\frac{1}{2})$. **3.** $g(-2)$. **4.** $g(3)$.

If f is the function defined by $y = 6 - x^2$, find

5. $f(-2)$. **6.** $f(-1)$. **7.** $f(0)$. **8.** $f(1)$.

If f is a function defined by $f(x) = x^2 - 2x + 3$, find

9. $f(0)$. **10.** $f(-1)$. **11.** $f(-3)$. **12.** $f(5)$.

Draw the graphs of the relations defined below, and give the domain and range of each relation in Exercises 17–24. State whether or not the relation is a function.

13. $f(x) = 5x - 1$; $-2 < x \leq 1$ **14.** $h(x) = |x|$; $-5 \leq x \leq 5$

15. $g(x) = \frac{1}{3}x - 2$; $-3 \leq x \leq 3$ **16.** $t(x) = -|x|$

17. $f(x) = \begin{cases} 3 \text{ when } x < 0 \\ -3 \text{ when } x > 0 \end{cases}$ **18.** $f(x) = \begin{cases} 1 \text{ when } x > 0 \\ 0 \text{ when } x = 0 \\ -1 \text{ when } x < 0 \end{cases}$

19. $g(x) = x - 2$ *or* $g(x) = x$ **20.** $f(x) = -x$ *and* $t(x) = x$

21. $f(x) = x^2 + 5x + 6$ **22.** $g(x) = x^2 + 11x + 30$

23. $f(x) = x^2 - 2x - 3$ **24.** $h(x) = 2x^2 - 5x + 2$

25. $f(x) = -2x^2 - 2x - 12$ **26.** $g(x) = 6x^2 + 9x - 6$

27. Describe the graph of $y = mx + b$ when m is 0; when b is 0; when both m and b are 0.

28. As $|a|$ increases, what happens to the graph of $y = ax^2$?

29. Would the graph of $y = -\frac{1}{2}x^2$ be inside or outside the graph of $y = -x^2$?

30. What point do all graphs of $y = ax^2$ for different values of a have in common?

31. What is the slope and y intercept of the linear function whose graph passes through $(-2, -3)$ and $(4, 4)$?

32. Assuming that the domain of the function in Exercise 31 is {real numbers}, what is its range?

33. Determine m and b if the graph of $f(x) = mx + b$ is the line segment joining $(-5, 6)$ and $(3, 4)$.

Determine the equation that defines the linear function whose graph passes through the points whose coordinates are as follows.

34. $(0, 0)$ and $(1, 2)$ **35.** $(5, 2)$ and $(3, -7)$

36. $(-1, 2)$ and $(3, 4)$ **37.** $(-3, -5)$ and $(2, -4)$

38. What is the y intercept of the graph of $y = x^2 + 3$?

39. What is the y intercept of $y = x^2 - 3$?

40. For any given value of x, is the point on $f(x) = x^2 - 1$ one unit lower than on $f(x) = x^2$? Explain.

B — If $f(x) = 3x$, find x if

41. $f(x) = 3$. **42.** $f(x) = 6$. **43.** $f(x) = -9$. **44.** $f(x) = -3$.

— If t is the function defined by $t(x) = x^2 - 2x - 20$, find

45. $t(a)$. **46.** $t(c)$. **47.** $t(a - c)$. **48.** $t(2a)$.

— If $f(x) = 2x + 1$, find x if

49. $f(x) = 0$. **50.** $f(x) = 1$. **51.** $f(x) = -1$. **52.** $f(x) = -2$.

— If h is a function defined by $y = |x|$, find x if

53. $h(x) = 2$. **54.** $h(x) = -2$. **55.** $h(x) = 0$. **56.** $h(x) = k$.

— Draw the graph of each of the following equations and state whether or not the equation defines a function.

57. $f(x) = 1$; $\frac{1}{2} \leq x \leq 2$

58. $f(x) = \begin{cases} 2 \text{ when } 1 \leq x \leq 3 \\ 1 \text{ when } 1 \leq x \leq 2 \end{cases}$

59. $f(x) = |x|\ (-5 \leq x \leq 5)$ *or* $f(x) = -x(-3 \leq x \leq 3)$

60. $f(x) = \begin{cases} -2 \text{ when } -3 \leq x < 3 \\ x \text{ when } -2 \leq x \leq 2 \\ x + 1 \text{ when } 0 \leq x < 1 \end{cases}$

61. Explain why moving a vertical line across the graph of a relation shows whether or not it is a function.

62. How do you obtain the range from the graph of the function? How do you obtain the domain from the graph of the function?

63. If f is a function that maps each real number x into the real number $x^2 + 5$, list ten ordered pairs in the function. Graph f.

64. The function g maps any real number r into $(r + 1)(r - 1)$. Draw the graph of the function.

C — If $f(x) = 2x^2 + 3x - 6$ and $g(x) = x^2 + 4$, find

65. $f(x) + g(x)$. **66.** $f(x) - g(x)$.

67. $g(x) - f(x)$. **68.** $f(x) + 2g(x)$.

69. $\frac{f(x)}{g(x)}$. **70.** $\frac{g(x)}{f(x)}$.

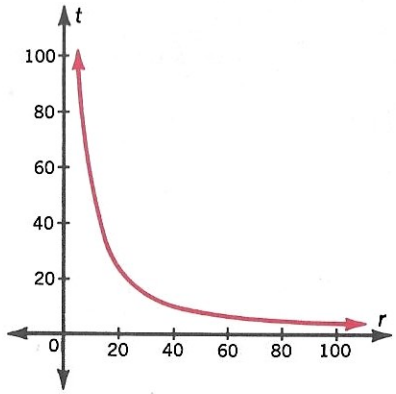
71. $f(g(x))$. **72.** $f(f(x))$.

73. $2f(x) - g(x)$. **74.** $3f(x) + 2g(x)$.

11–4 Inverse Variation

You know that the faster the speed, the less time it takes to make a trip. For a trip of 480 miles, the relationship between rate of travel (speed) and time can be shown by the equation $rt = 480$. The table below shows six number pairs that satisfy this relation.

r	80	60	40	30	20	10	5
t	6	8	12	16	24	48	96



Is the relation a function? Notice that the domain can include only positive numbers. What is the range? As the values for t increase, do the values for r increase or decrease? As the values for r increase, do the values for t increase or decrease?

In Chapter 4 you learned that the function of *direct variation*, $y = kx$, has the following properties. As the values for x increase, the values for y increase, and as the values for x decrease, the values for y decrease. Every value for y is a constant times every value for x. Clearly, the relation $rt = 480$ exhibits a kind of variation that is the opposite of direct.

When the product of two variables is a nonzero constant, an increase in the values of one variable will cause a decrease in the values of the other one. Also, a decrease in the values of one variable will cause an increase in the values of the other one.

Definition Inverse variation is a variation of two related variables such that the product of the variables is a nonzero constant. The function of inverse variation may be defined by

$$y = \frac{k}{x},\ k \neq 0,$$

or equivalently by

$$xy = k,\ k \neq 0.$$

The constant k is the constant of variation.

The equation $rt = 480$ implies that **r varies inversely as t.** As a function, you might write $f(t) = r = \frac{480}{t}$. Notice that it is also true that t varies inversely as r, since $t = \frac{480}{r}$ is true.

Often it is convenient to express direct variations in terms of a proportion. For example, for two pairs of numbers (x_1, y_1) and (x_2, y_2) that satisfy $y = kx$, you can write the following true statements.

(1) $y_1 = kx_1$, or $\frac{y_1}{x_1} = k$ (2) $y_2 = kx_2$, or $\frac{y_2}{x_2} = k$

Then, because each left side equals k, you get the following proportion.

$$\frac{y_1}{x_1} = \frac{y_2}{x_2}$$

Similarly, there is a connection between proportions and inverse variation. Consider (x_1, y_1) and (x_2, y_2) that satisfy $xy = k$.

Upon substituting, the following statements are true.

(1) $x_1y_1 = k$ (2) $x_2y_2 = k$

Therefore, $x_1y_1 = x_2y_2$ is true. Why? This leads to $\frac{x_1}{x_2} = \frac{y_2}{y_1}$. Why?

Exercises

A — Use the equation $rt = 300$ to complete the following sentences.

1. The formula $rt = 300$ is an example of ? (direct, inverse) variation.

2. When t is 5, r is ?. When t is 10, r is ?. When a value of t is doubled, the value of r is ?.

3. When t is 15, r is ?. When t is $7\frac{1}{2}$, r is ?. When a value of t is halved, the value of r is ?.

4. When r is 50, t is ?. When r is 100, t is ?. When a value of r is doubled, the value of t is ?.

5. When r is 50, t is ?. When r is 25, t is ?. When a value of r is halved, the value of t is ?.

— Mary has \$3.00 to spend for candy and she always spends the whole amount. There is a relation between the number of pieces, n, she can buy and the cost per piece, c, in cents. The relation can be shown by the equation $nc = 300$. Use this relation to complete the following sentences.

6. If c is 2, n is ?. If c is 6, n is ?. If the cost per piece is tripled, the number she can buy is ?.

7. If c is 12, n is ?. If c is 3, n is ?. If the cost per piece is divided by 4, the number of pieces she can buy is ?.

8. If n is 10, c is ?. If n is 50, c is ?. If the number she buys is multiplied by 5, the cost per piece is ?.

9. If n is 20, c is ?. If n is 5, c is ?. If the number she buys is divided by 4, the cost per piece is ?.

10. Does c vary inversely as n? Does n vary inversely as c?

11. Which of the following are examples of inverse variation?

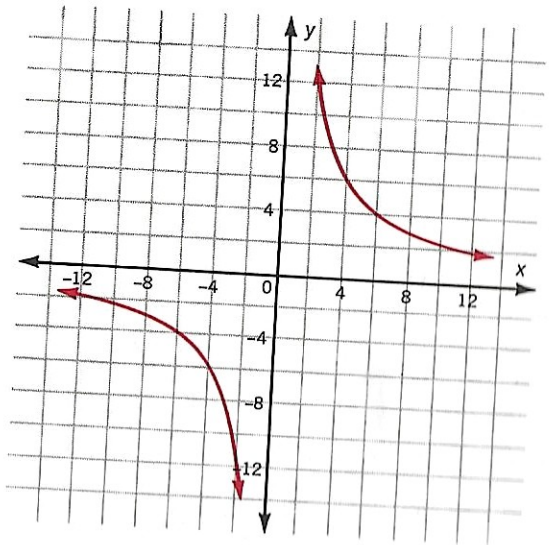
a. $xy = 6$ **b.** $x + y = 6$ **c.** $x - y = 6$

d. $\frac{x}{y} = 6$ **e.** $x = \frac{10}{y}$ **f.** $x = 10y$

g. $x - y = 10$ **h.** $x + y = 10$ **i.** $y = 250x$

j. $y = 430x - 5$ **k.** $xy + y = 8$ **l.** $y = \frac{42}{x}$

12. Shown below is the graph of $xy = 24$ when the replacement set for x and y is R. Explain why there are two curves in the graph. Why is $xy = 24$ the equation of a function? When you think of the function as $f(x) = \frac{24}{y}$, explain why 0 is in neither the domain nor the range.



13. Draw a graph for each example of inverse variation in Exercise 11. Are the graphs lines or are they some other kind of curve?

14. There is an infinite number of rectangles that have an area of 400 square feet. Write an equation that shows the relation be-

tween the length, l, and the width, w, of rectangles with an area of 400 square feet.

— Use the equation from Exercise 14 to complete the following.

15. If l increases, then w ?.

16. If l decreases, then w ?.

17. If a certain value of l is doubled, the value of w is ?.

18. If a certain value of w is multiplied by 3, the value of l is ?.

19. If a certain value of l is multiplied by $\frac{1}{2}$, the value of w is ?.

— If you invest \$1000, you might be interested in the length of time and the rate of interest required to earn \$200 on the investment. The equation $200 = 1000rt$ expresses the relation between r, the rate of interest per year, and t, the time in years the \$1000 is invested. Use this formula for Exercises 20–25.

20. Make a table showing the pairs of numbers for r and t when t is any member of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

21. To earn \$200 in 5 years, the rate of interest must be ?.

22. To earn \$200 in 1 year, the rate of interest must be ?.

23. Use the ten number pairs for r and t from Exercise 20 to graph ten points of the relation. Do the points lie along a straight line? Sketch a smooth curve containing the points.

24. If the number of years required to earn \$200 increases, does the rate of interest increase or decrease?

25. If the rate of interest increases, does the time increase or decrease?

26. If y varies inversely as x, and y is 12 when x is 4, what is the constant of variation? Write the equation that shows the relation between x and y.

27. For rectangles of the same area, the base varies inversely as the height. What is the constant of variation when the base is 10 feet and the height is 12 feet? Write the equation that shows the relation between base and height for rectangles that have this area.

— If y varies inversely as x, find the constant of variation and write the equation showing the relation between x and y for each of the following.

28. $y = 7$ when $x = 6$

29. $y = 4$ when $x = 12$

30. $y = \frac{3}{4}$ when $x = \frac{2}{3}$

31. $y = -5$ when $x = 2$

32. $y = -8$ when $x = -5$

33. $y = -5\frac{1}{4}$ when $x = -8\frac{1}{3}$

— Each of these equations is an example of inverse variation.

a. $xy = 10$ **b.** $xy = \frac{1}{2}$ **c.** $x = \frac{12}{y}$ **d.** $xy = .01$

34. Find two number pairs that satisfy each equation.

35. For each equation, find the ratio of the x numbers of the pairs and the ratio of the y numbers of the pairs.

36. Are the two ratios equal?

37. Is one of the ratios equal to the multiplicative inverse of the other one?

B — Which of the following are examples of direct variation and which are examples of inverse variation?

38.

x	−4	−2	0	2	4	6
y	−12	−6	0	6	12	18

39.

x	−5	−3	−1	1	3	5
y	15	9	3	−3	−9	−15

40.

x	−12	−6	−2	−1	1	2	4	6	12
y	−2	−4	−12	−24	24	12	6	4	2

41.

x	−5	−2	1	4	7	10
y	3	6	9	12	15	18

42.

x	−10	−4	0	4	8	12
y	$-7\frac{1}{2}$	−3	0	3	6	9

43.

x	−5	−2	0	3	4	5	10
y	50	8	0	18	32	50	200

44. $P = 4s$ The perimeter of a square equals 4 times the length of a side.

45. $t = \frac{400}{r}$ To travel 400 miles at r miles per hour requires t hours.

46. $d = 50t$ The distance traveled at 50 miles per hour equals 50 times the number of hours.

47. $f = 3y$ The number of feet in any length equals 3 times the number of yards.

48. $AR = 1000$ The amount of the taxable value of property times the tax rate equals \$1000 in taxes.

49. $l = 6e$ The length of the 6 edges of any cube equals 6 times the length of any edge.

50. $b = \frac{96}{a}$ The base of any rectangle with an area of 96 square inches equals 96 divided by the altitude of the rectangle.

51. $C = 2\pi r$ is the formula for the circumference of a circle in terms of its radius. Complete: The circumference of a circle varies ? (directly, inversely) as its radius. What is C if r is 10? 100?

If y varies inversely as the square of x, then $y = \frac{k}{x^2}$, or $yx^2 = k$, is true. Find the constant of variation and write the equation showing the relation for each of the following.

52. $y = 6$ when $x = 2$

53. $y = 2$ when $x = 1$

54. $y = 8$ when $x = 4$

55. $y = 4$ when $x = -3$

56. $y = -5$ when $x = -4$

57. $y = 3\frac{1}{2}$ when $x = 5\frac{3}{4}$

58. The formula for the area of a circle is $A = \pi r^2$. Complete this table showing pairs of numbers for r and A. A varies *directly as the square* of r.

r	1	2	3	4	5	6	7	8	9	10
A	π	4π	9π	16π	?	?	?	?	?	?

Use the results of Exercise 58 to complete the following.

59. If any number for r is doubled, the number for A is ?.

60. If any number for r is multiplied by 5, the number for A is ?.

61. A circle with a radius of $7\frac{1}{2}$ inches has an area ? times the area of a circle with a radius of 15 inches; ? times the area of a circle with a radius of 30 inches.

62. If an object is dropped from some height, the approximate distance it will fall in any number of seconds is expressed by the formula $d = 16t^2$. Complete the following table showing the relation between d and t. Does d vary directly as t, inversely as t, or directly as the square of t?

t	1	2	3	4	5	6
d	16	64	144	?	?	?

Use the results of Exercise 62 to complete the following.

63. If t is doubled, d is ?.

64. If t is tripled, d is ?.

65. If t is halved, d is ?.

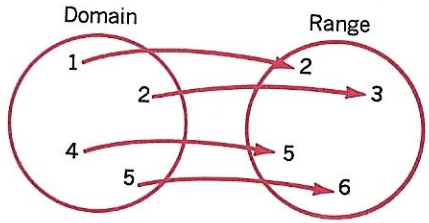
C **66.** The formula for the volume of any sphere is $V = \frac{4}{3}\pi r^3$. You can say of a variation such as this that the volume varies *directly as the cube* of the radius. Complete: If r is doubled, V is ?. If r is halved, V is ?.

67. The frequency of vibration of a string (of given length and size) varies *directly as the square root* of the tension (pounds of pull) on the string. This may be written as the formula $f = k\sqrt{t}$. If k is 60, what is f when t is 49? when t is 25?

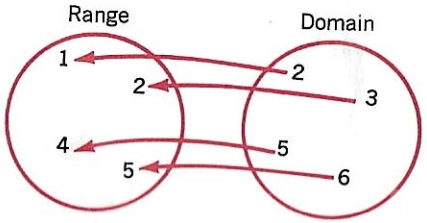
68. The frequency of vibration of a string (uniform length and tension) varies *inversely as the square root* of its weight. Express this relation in a formula, using k to represent a constant. Given that k is 508, what is f when w is 4?

11–5 Inverses of Relations and Functions

Consider the relation S = {(1, 2), (2, 3), (4, 5), (5, 6)}. The domain of S is {1, 2, 4, 5}. What is the range? Note that the relation may also be viewed as a mapping.



In any mapping, the rule may be reversed to give another mapping. For example, to read a road map, you reverse the rule by using the map to picture the physical world. To reverse the rule for S, you merely need to put the arrowheads at the other end of the curves. The domain and range are then interchanged.



The result may then be written as the relation

{(2, 1), (3, 2), (5, 4), (6, 5)}.

Because of the way the relation $\{(2, 1), (3, 2), (5, 4), (6, 5)\}$ was formed from S, it is called the **inverse** of S. It is symbolized as S^{-1} and read "S inverse." Observe that S^{-1} is the relation whose members were obtained by interchanging the coordinates of each ordered pair of S.

Definition The inverse of a relation S is a relation S^{-1} that is obtained by interchanging the coordinates of every ordered pair of the given relation S.

Since functions are relations, the inverse of a function may be given the same definition. The inverse of f is f^{-1}. Notice that by interchanging coordinates, the domain of f becomes the range of f^{-1} and the range of f becomes the domain of f^{-1}.

The inverse of a function is not always a function itself. Explain why *the inverse of a function is also a function if and only if no horizontal line intersects the graph of the function in more than one point.*

When a function is defined by an equation, such as $y = 2x$, the equation of the inverse can be found by interchanging x and y. This has the effect of interchanging the coordinates. Then solve for y to represent the function in $y = f(x)$ form as follows.

	$y = 2x$
Interchange x and y.	$x = 2y$
Solve for y.	$y = \frac{1}{2}x$

You can show that $y = \frac{1}{2}x$ is the inverse of $y = 2x$ by comparing the solutions. Typical solutions of $y = 2x$ are (1, 2), (3, 6), and (4.5, 9). When the coordinates of these pairs are reversed, they become (2, 1), (6, 3), and (9, 4.5), which are solutions of $y = \frac{1}{2}x$.

A relation and its inverse may also be compared by looking at their graphs, shown in Figure 1.

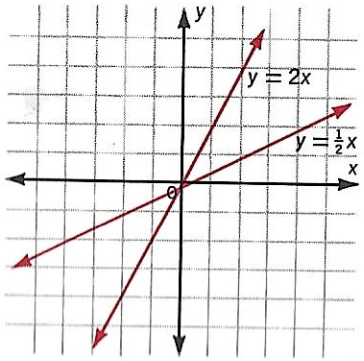


Figure 1

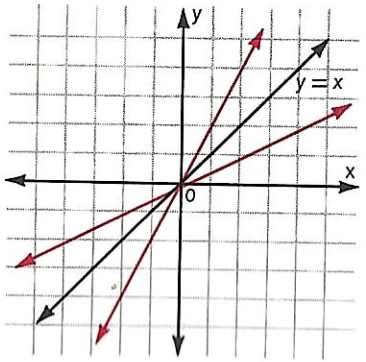
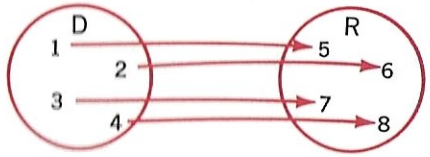


Figure 2

When you look at the two graphs in Figure 1 in conjunction with the graph of $y = x$, Figure 2, an important and useful relationship becomes apparent. If the graph were folded along the line defined by $y = x$, the relation and its inverse would coincide. The graph of R^{-1} is the reflection of the graph of R in relation to the graph of $y = x$. This is true for the graphs of any relation and its inverse.

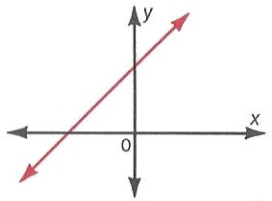
Exercises

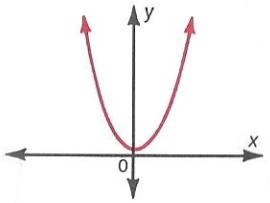
A — Give the inverse of each of the following relations.

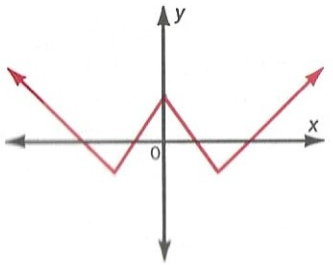
1. $R = \{(1, 3), (2, 4), (3, 6), (4, 8)\}$
2. $S = \{(1, 0), (2, 0), (3, 0), (4, 0), (5, 0)\}$
3. $T = \{(1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (1, 5)\}$
4. $U = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$
5. $V = \{(1, 2), (2, 1), (3, 4), (4, 3), (5, 6), (6, 5)\}$
6. 

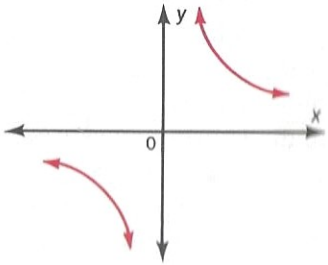
7. $f = \{(x, y) : y = 3x\}$
8. $g = \{(x, y) : y = 2x + 1\}$
9. $h = \{(x, y) : y = -4x - 5\}$
10. $F = \{(x, y) : y = x^2\}$

B — Which of the functions graphed below have inverses that are functions?

11. 

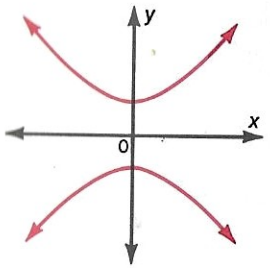
12. 

13. 

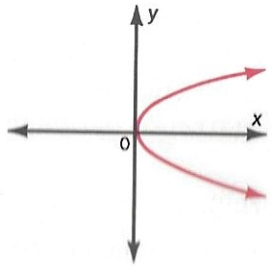
14. 

— Which of the relations graphed below that are not functions have inverses that are functions?

15.



16.



— What are the coordinates of the reflections of the following points if the points are reflected through the line $y = x$?

17. (4, −2) **18.** (−2, 6) **19.** (3, 3) **20.** (3, −5)

— Find and graph the ordered pairs of each function. Graph the inverse of each function by reflecting each point through the line $y = x$.

21. $y = x$, $x \in \{1, 2, 3\}$ **22.** $y = 2x$, $x \in \{2, 4, 6\}$

23. $y = -x$, $x \in \{1, 3, 5\}$ **24.** $y = x + 1$, $x \in \{0, 1, 2, 3\}$

25. Find the equation that defines the inverse of each of the functions in Exercises 21–24.

C **26.** Does the set $\{(1, 2), (3, 2), (2, 1)\}$ define a function? Is it one-to-one? One-to-many? Is its inverse a function?

27. Does the set $\{(1, 2), (2, 3), (3, 4)\}$ define a function? Is it one-to-one or one-to-many? Is its inverse a function?

28. From Exercises 26 and 27, can you tell what kind of function has an inverse that is also a function?

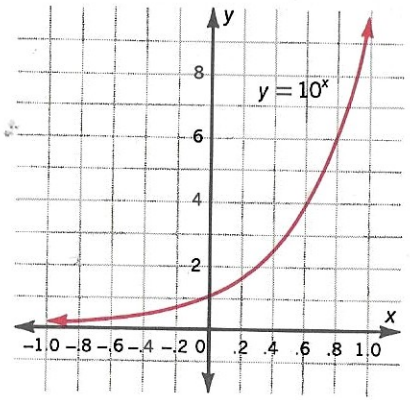
11–6 Exponential Functions

You have previously studied exponents and can find the decimal place-value representation of such expressions as 3^2, 5^{-1}, and $16^{\frac{1}{2}}$. Any real number can be used as an exponent, although you have learned only how to use the rational numbers so far. Thus, it is possible to find a value of an expression such as 10^x for any real replacement for x. For any real number x, a solution of $y = 10^x$ may be obtained. There is only one value of y for each value of x. Hence, the equation $y = 10^x$ defines a function, known as an **exponential function.**

The table below shows several pairs of numbers that satisfy $y = 10^x$. Approximations are given for y when x is not an integer.

x	−3	−2	−1	0	0.2	0.4	0.6	0.8	1	1.2	1.4
y	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$	1	1.6	2.5	4.0	6.3	10	16	25

The solutions from the table can be plotted to form a partial graph of the function. By drawing a smooth curve through these points, you obtain the graph of the function.



Look at the graph and answer these questions.

a. As the values for x increase, do the values for y increase or decrease?

b. As the values for x decrease, do the values for y increase or decrease?

c. Do you think that the curve crosses the x axis?

d. At what point does the curve cross the y axis?

Notice that although the values of y increase and decrease along with the values of x, this is not the same as direct variation. In this case they do not increase or decrease at proportional rates. In fact, the values for y increase very much for large values of x, but very little for small ones. The curve crosses the y axis at (0, 1) because $1 = 10^0$. The curve will not cross the x axis because $0 = 10^x$ has no solution.

Other exponential functions can be obtained by substituting another number for 10. For example, $y = 2^x$, $y = e^x$ (where $e \approx 2.71828$), and $y = 3^x$ all define exponential functions.

Definition An exponential function is a function defined by $f(x) = b^x$, where the number b, called the base, is greater than zero but not equal to one.

Exercises

A

1. What is the domain and range of $y = 10^x$? Use the table of values given at the beginning of this section.

Graph the following exponential functions and draw smooth curves through each set of points, $x \in R$.

2. $y = e^x$ Use this table of values.

x	−3	−2	−1	0	1	2	3
y	0.05	0.14	0.37	1	2.72	7.39	20.1

3. $y = 2^x$ Complete the table and use these values.

x	−4	−3	−2	−1	0	1	2	3	4	5	10
y	0.06	0.125	0.25	?	?	?	?	?	?	?	?

4. $y = 3^x$ Complete the table and use these values.

x	−4	−3	−2	−1	0	1	2	3	4
y	?	?	?	?	?	?	9	27	81

5. $y = 5^x$

B

6. Is it true that no exponential function $y = b^x$ crosses the x axis?

Exercises 7–11 refer to $y = 10^x$.

7. As x increases from 0 to 1, y increases from ? to ?.

8. As x increases from 1 to 2, y increases from ? to ?.

9. Is the rate of increase of y in an interval greater than or less than the corresponding increase in x for that interval?

10. Does the graph appear to get steeper as x increases?

11. Will the graph eventually reach a point where it stops curving and becomes a straight line? Explain.

12. In the definition of an exponential function, $y = b^x$, the restriction $b \neq 1$ is made. Suppose b is 1. Then what kind of function is defined? Graph this function.

13. Graph $y = 10^x$ and $y = (\frac{1}{10})^x$ on the same set of axes.

Use the graphs in Exercise 13 to answer the following questions.

14. Are both graphs smooth curves?

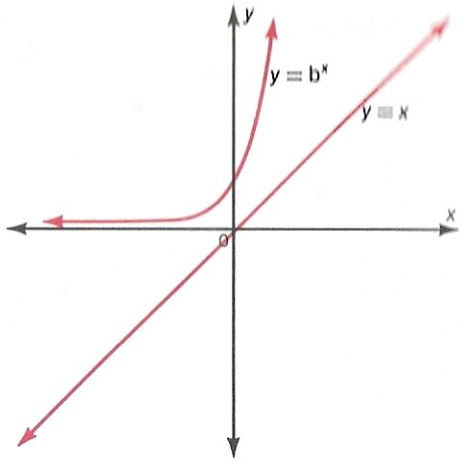
15. When $b > 1$ and x increases, does y increase or decrease?

16. When $0 < b < 1$ and x increases, does y increase or decrease?

17. Do the graphs show points in common? If so, which ones?

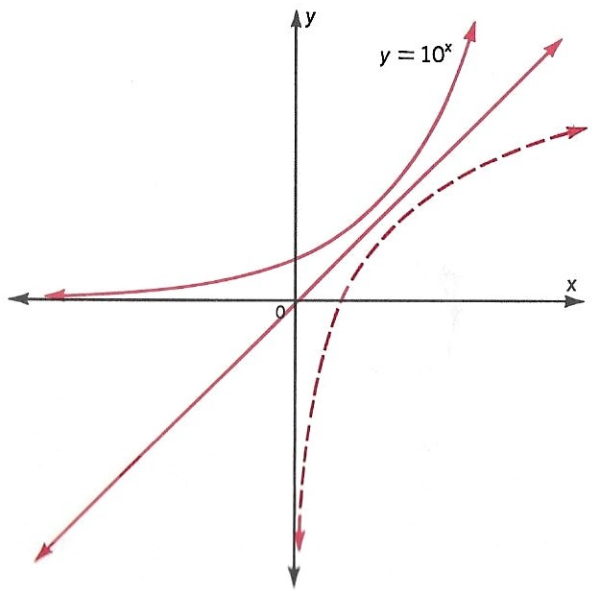
18. Describe the domain and range of the graph of $y = (\frac{1}{10})^x$.

19. Draw the reflection of the graph of $y = b^x$, $b > 1$, if the reflection is through the line $y = x$. At what point does the reflected curve intersect the x axis? What is the relationship between $y = b^x$ and the function whose graph is the reflected curve?



11–7 Logarithmic Functions

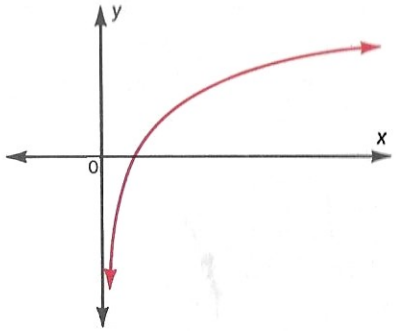
In the figure below, the dashed curve is the reflection of the graph of $y = 10^x$ in the graph of $y = x$. The dashed curve represents the inverse of $y = 10^x$. Since for each value for x, there is exactly one value for y, the inverse is a function also. The inverse is a **logarithmic function.** It is defined by $x = 10^y$, or by $y = \log_{10} x$ (read "y equals the logarithm of x to the base ten" or "y equals log base ten x").



Exponential function	*Logarithmic function*
$y = 2^x$	$x = 2^y$, or $y = \log_2 x$
$y = e^x$ ($e \approx 2.71828$)	$x = e^y$, or $y = \log_e x$
$y = 3^x$	$x = 3^y$, or $y = \log_3 x$

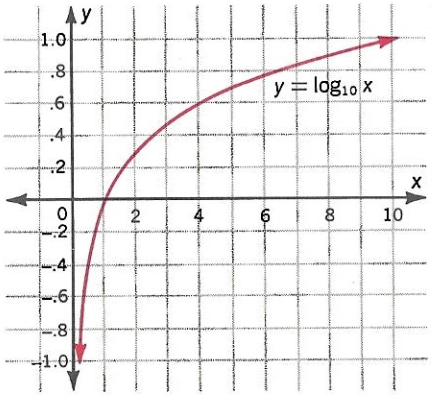
Definition A logarithmic function is a function defined by $y = \log_b x$, which means $x = b^y$, where b is greater than zero and not equal to one. The value of b is the base of the function.

For b greater than 1, the graph is similar to the one at the right. Notice that as the values for x increase, so do the values for y. Similarly, as the values for x decrease, so do the values for y. What is the x intercept? Based upon your work with exponential functions, do you think there is a y intercept?



Exercises

A — Use the graph of $y = \log_{10} x$ to answer Exercises 1–7.



1. When x is greater than 1, is the logarithm of x a positive or a negative number?

2. When x is between 0 and 1, is the logarithm of x a positive or a negative number?

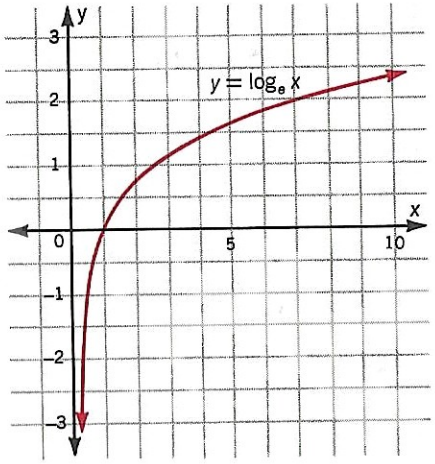
4. As x gets closer and closer to 0, what happens to the graph?

5. When x equals 10, what is the value of y?

6. What is the domain of $y = \log_{10} x$?

7. What is the range of $y = \log_{10} x$?

B — Use the graph of $y = \log_e x$ ($e \approx 2.71828$) to answer the following questions.



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8. When x is greater than 1, is the logarithm of x a positive or a negative number?

9. When x is 1, what is $y = \log_e x$?

10. When x is between 0 and 1, is the logarithm of x a positive or a negative number?

11. Can x equal 0? Why or why not?

12. What is the domain of $y = \log_e x$?

13. What is the range of $y = \log_e x$?

— From the graphs of $y = \log_{10} x$ and $y = \log_e x$, certain common properties are apparent for the general logarithmic function $y = \log_b x$, $b > 1$. Complete the following for that function.

14. As x increases, y ?. As x decreases, y ?.

15. The domain of $y = \log_b x$ is {positive real numbers} and the range is ?.

16. When x is greater than 1, the logarithm of x is ? (positive, negative, or zero).

17. When x equals 1, the logarithm of x is ? (positive, negative, or zero).

18. When $0 < x < 1$ is true, the logarithm of x is ? (positive, negative, or zero).

19. When x equals the base b, y equals ?.

20. The graph of $y = \log_b x$, $b > 1$, gets closer and closer to the ? axis, but never crosses it.

C — Use the table of values prepared below to draw the graphs of the given function. Draw a smooth line through the points. Check with the graphs drawn in parts A and B of this exercise set.

21. $y = \log_{10} x$

x	1	2	3	4	5	6	7	8	9	10
y	0	0.30	0.48	0.60	0.69	0.77	0.84	0.90	0.95	1.00

22. $y = \log_e x$

x	1	2	3	4	5	6	7	8	9	10
y	0	0.69	1.09	1.38	1.60	1.79	1.94	2.07	2.19	2.30

11–8 Computation with Logarithms

The value of a logarithmic function for a particular replacement for x is called the **logarithm** of the replacement. Logarithms are exponents; the value of $y = \log_{10} x$ is the number y such that $x = 10^y$ is true.

For every property of exponents, there is a corresponding property of logarithms. The following table shows how these properties are related for the operations of multiplication, division, and raising to a power.

Operation	Exponents	Logarithms
Multiplication	$a^m \cdot a^n = a^{m+n}$ (Add exponents.)	$\log xy = \log x + \log y$ (Add logarithms.)
Division	$\frac{a^m}{a^n} = a^{m-n}$ (Subtract exponents.)	$\log \frac{x}{y} = \log x - \log y$ (Subtract logarithms.)
Raising to a power	$(a^m)^n = a^{mn}$ (Multiply exponents.)	$\log x^y = y \log x$ (Multiply the logarithm by the power.)

State in words the properties of the logarithm of a product, of a quotient, and of a power.

Use of these properties simplifies computation, as can be seen from the following examples.

EXAMPLE 1. Use logarithms to find 3.25×1.63.

Since 10 is the base of the numeration system, it is convenient to use logarithms to base 10. A table of logarithms to base 10, called **common logarithms,** is provided on pages 548 and 549.

Find log 3.25 in the table. First locate 3.2 in the column headed N. In this same row, in the column headed 5, you will find 5119. Notice that the columns represent different hundredths places and that the decimal points are omitted for each logarithm. Since 3.25 is between $10^0 = 1$ and $10^1 = 10$, log 3.25 is between 0 and 1; so 5119 means 0.5119. Using the table in a similar way, you can also determine that log 1.63 is approximately 0.2122. Now apply the property of the logarithm of a product.

$$\begin{aligned} \log 3.25 &\approx 0.5119 \\ \log 1.63 &\approx 0.2122 \\ \hline \text{Add the logarithms.} \qquad & 0.7241 \end{aligned}$$

The logarithm of the product is approximately 0.7241. Use the table to find the logarithm nearest 0.7241. It is 0.7243. The entry 7243 is in the 5.3 row and the 0 column. Hence, 0.7243 is the approximate logarithm of 5.30.

Therefore, 3.25×1.63 is approximately 5.30. For most purposes the error caused by the various approximations is so small that it can be ignored. For example, the exact product of 3.25 and 1.63, found in the ordinary way, is 5.2975.

EXAMPLE 2. Use logarithms to find $\frac{8.75}{2.36}$.

$$\begin{aligned} \log 8.75 &\approx 0.9420 \\ \log 2.36 &\approx 0.3729 \\ \hline \text{Subtract the logarithms.} \qquad & 0.5691 \end{aligned}$$

The difference, 0.5691, is the logarithm of the quotient. In the table, 0.5691 corresponds most nearly to the number 3.71. Therefore,

$$\frac{8.75}{2.36} \approx 3.71$$

is true.

EXAMPLE 3. Find 1.27^4.

Use the property of powers.

$\log 1.27 \approx 0.1038$

$\log 1.27^4 = 4 \log 1.27$

$4 \log 1.27 \approx 0.4152$

$1.27^4 \approx 2.60$

Exercises

A — Use the table to find the common logarithms of the following numbers.

1. 3.92	**2.** 1.31	**3.** 3.21	**4.** 5.06
5. 2.01	**6.** 7.91	**7.** 9.00	**8.** 4.59

— Perform the indicated computation using logarithms.

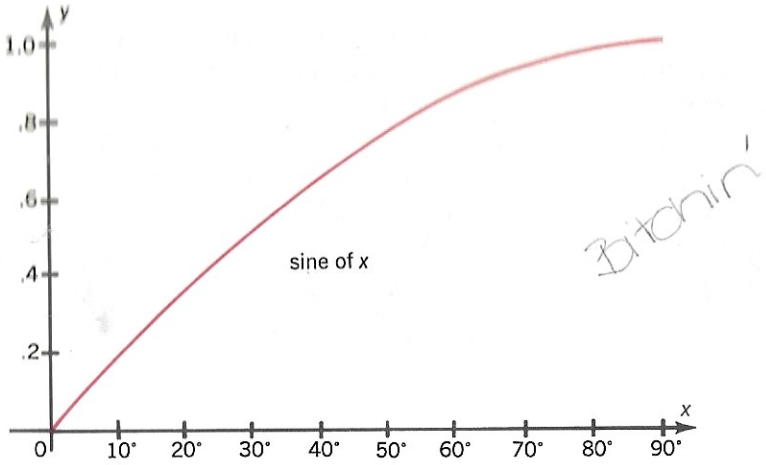
9. 3.92×1.31	**10.** $\frac{5.06}{2.01}$
11. $(3.21)^2$	**12.** 5.21×1.63
13. 2.32×2.54	**14.** $(1.21)^5$
15. 3.23×2.62	**16.** $(3.51)(1.82)$
17. $\frac{(7.91)}{(3.82)}$	**18.** $(1.23)^3$
19. $(1.01)^4$	**20.** $\frac{8.71}{2.83}$
21. $\frac{9.00}{3.78}$	**22.** $(1.11)^8$
23. $(2.65)(3.27)$	**24.** 4.86×1.01

11–9 Trigonometric Functions

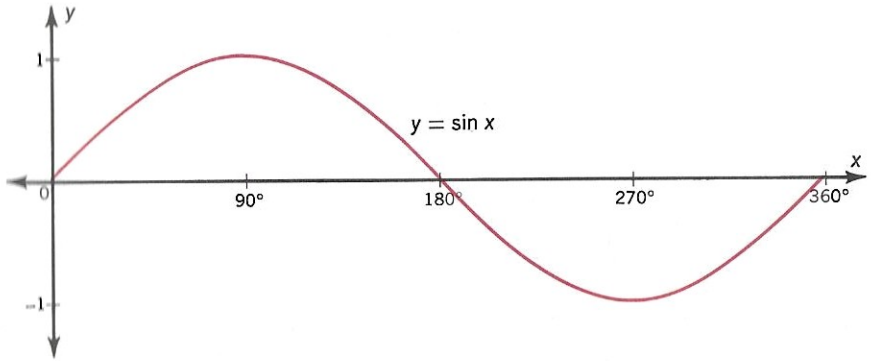
The equation $y = \sin x$ shows the relation between a number of degrees, x, and the ratio of the measure of the side opposite an acute angle in a right triangle to the hypotenuse. This ratio is y, the sine of the angle. In the table of values of the trigonometric functions, you may think of the column headed *Angle* as x and the column headed *Sine* as y. For each angle x, is there one and only one number for y? There is only one such number, so $y = \sin x$ defines a function. Some of the number pairs are shown in the table below.

When x is	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
Then y is	.000	.1736	.3420	.5000	.6428	.7660	.8660	.9397	.9849	1.0000

The graph of $y = \sin x$ is shown below ($0° \leq x \leq 90°$).



The relationship between the units on the scales is chosen so that one unit on the vertical axis is approximately equivalent to 57° on the horizontal axis. The reasons for this will become apparent in later courses. As you study trigonometric functions further, you will extend the domain of the sine function to the full set of real numbers. For $\{x : 0° \leq x \leq 360°\}$, the graph of $y = \sin x$ looks like the following.



Exercises

A — Use the table of values on page 547 in these exercises.

1. Complete this table showing ten number pairs that satisfy the equation $y = \cos x$.

When x is	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
y is	1.0000	.9848	?	?	?	?	?	?	?	?

2. Draw a graph of the function defined by $y = \cos x$ when the domain is $0 \leq x \leq 90°$.

3. The equation $y = \tan x$ shows the relation between x, the number of degrees, and y, the tangent of x. From the table you can see that the tangent is undefined for an angle of 90° because it involves division by 0. Then, 90° is not a possible choice for x. For every other number for x, is there one and only one number for y? Does $y = \tan x$ define a function?

4. Complete this table showing ten number pairs that satisfy the equation $y = \tan x$.

When x is	0°	10°	20°	30°	40°	50°	60°	70°	80°	89°
y is	?	?	?	?	?	?	?	?	?	?

5. Draw a graph of $y = \tan x$ when the domain is $0° \leq x < 90°$. Notice that x cannot be 90°.

B

6. Complete this table showing ten number pairs that satisfy the equation $y = 3(\sin x)$. Each number for y is 3 times the value of $\sin x$.

When x is	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
y is	.0000	.5208	1.0260	?	?	?	?	?	?	?

7. Make a table and draw the graph of $y = 3(\sin x)$ when the domain is $0° \leq x \leq 90°$.

8. Make a table and draw the graph of $y = 3(\cos x)$ when the domain is $0° \leq x \leq 90°$.

CHAPTER REVIEW

1. Know the meaning of and be able to use each of the following words or phrases. The number shown after each word or phrase indicates where it is introduced, in case you need to review.

rule (*462*)
mapping (*463*)
domain (*463*)
range, or image (*463*)
one-to-one mapping (*464*)
many-to-one mapping (*464*)
one-to-many mapping (*464*)
inverse variation (*475*)
inverse of a relation (*482*)
exponential function (*485*)
logarithmic function (*488*)
logarithm (*490*)
common logarithm (*491*)
trigonometric function (*492*)

The perimeter P of a square is related to its side s by the equation $P = 4s$. Express this same relation by a graph, a table, a sentence, and a set of ordered pairs (s, P).

3. Consider the relation $R = \{(1, 2), (2, 3), (4, 5), (6, 7)\}$. What is the domain of the relation? What is the range?

4. If $f(x) = \frac{2x - 3}{x}$, find $f(-2)$ and $f(3)$.

5. Draw the graph of the function g defined by the following.

$$g(x) = \begin{cases} -3 \text{ when } x < 0 \\ 0 \text{ when } x = 0 \\ 2 \text{ when } x > 0 \end{cases}$$

6. Draw the graph of $f(x) = 3x - 2$ for the domain $-2 \leq x \leq 2$.

7. When $b = 0$ in $y = mx + b$, what kind of variation exists between x and y?

8. Express inverse variation as a proportion showing that y varies inversely as x.

9. What is the equation of the line about which a relation can be folded to get its inverse?

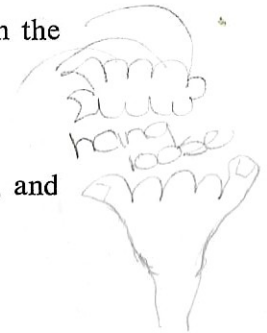
10. Given an equation that defines a function, how do you form the equation that defines the inverse of this function?

11. State *four* of the properties of the graph $y = b^x$.

12. State *four* of the properties of the graph $y = \log_b x$.

13. State the property of a logarithm of a product, a quotient, and a power.

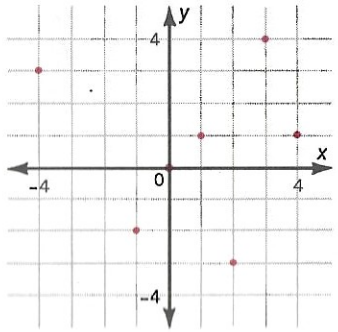
14. Define each trigonometric function you have studied.



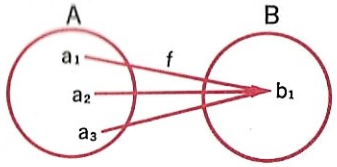
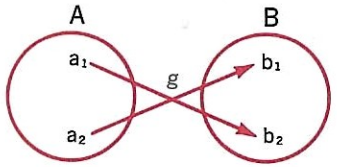
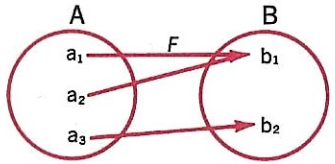
CHAPTER TEST

1. Define the relation $y = 2x$ by a table, a graph, and a set of ordered pairs.

2. Using the graph of the function f defined at the right, find the domain and the range of f.



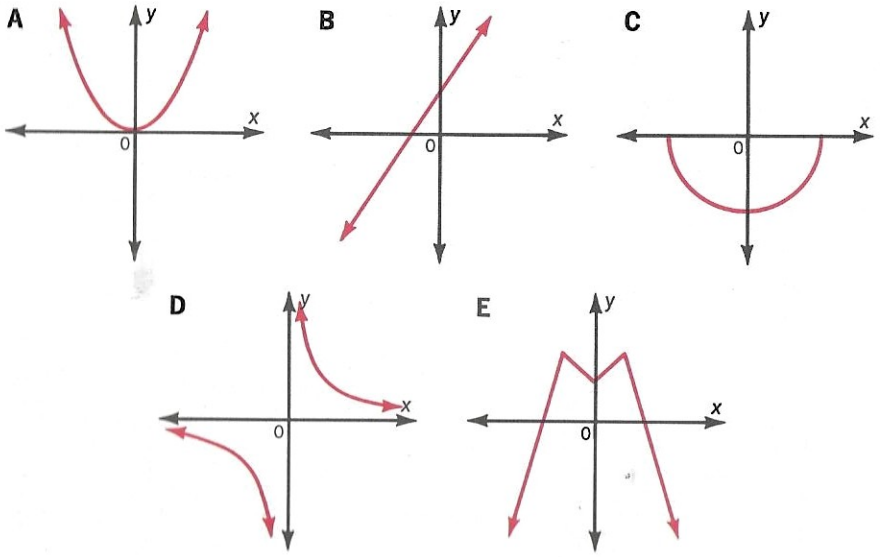
Each mapping in Column I illustrates a special kind of function. Find its name in Column II. More than one answer may be used.

Column I	Column II
3. 	a. One-to-one function.
4. 	b. Many-to-one function.
5. 	c. Constant function.

6. Determine m and b of $y = mx + b$ such that its graph passes through $(-2, 1)$ and $(3, 4)$.

7. If y varies inversely as x, and y is 30 when x is 6, write the equation that shows the relation between x and y.

8. Determine which of the functions graphed below have inverses that are functions.

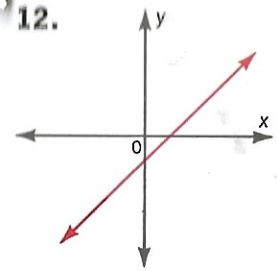


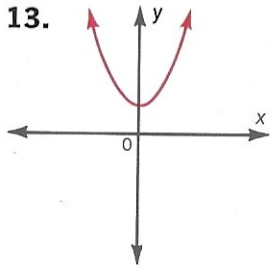
9. If $f(x) = 2x^3 - 3x + 1$, find $f(-1)$ and $f(t)$.

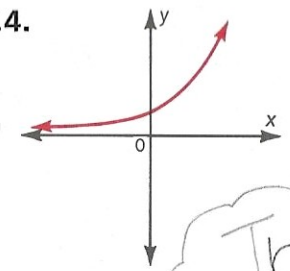
10. If $f = \{(1, 5), (-2, 3), (4, 6), (7, 8)\}$, then what is f^{-1}?

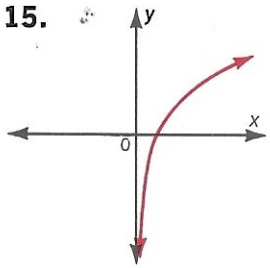
11. If $f(x) = 3x - 1$, determine the equation that defines its inverse.

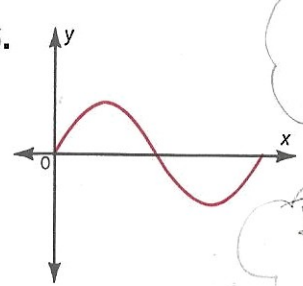
Name each function graphed below.

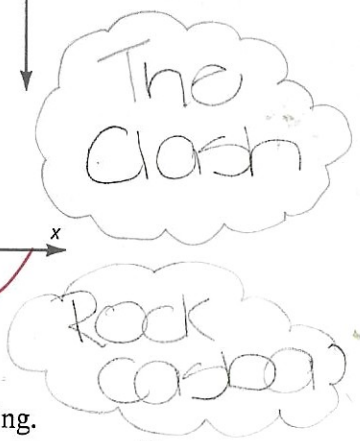
12.


13.


14.


15.


16.




If $\log 2 = 0.3010$ and $\log 3 = 0.4771$, find the following.

17. $\log (2 \cdot 3)$ **18.** $\log \frac{3}{2}$ **19.** $\log 3^2$ **20.** $\log \sqrt{2}$

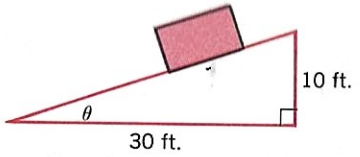
ENRICHMENT

Resolution of Forces Using Trigonometry

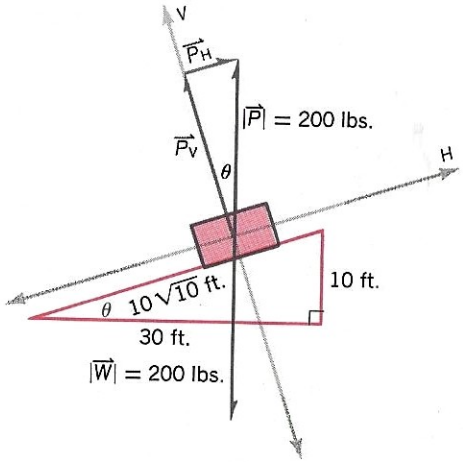
When vectors were presented after Chapter 4, you were dealing with the vector components of the force on an object. Seldom does an engineer measure his force vectors by their components; rather he measures the magnitude with a force gauge, and the direction is given by the configuration of the structure. Thus, the usual situation is that the force diagram will contain magnitudes and angles, or magnitudes and physical dimensions, from which he must determine the horizontal and vertical components by using trigonometry.

In analyzing a structure on which forces are exerted, the engineer uses a **free-body diagram.** A free-body diagram shows only the external forces felt by the body. If all the forces being considered operate on

point diagram. For example, suppose there is a box weighing 200 pounds, resting on a ramp inclined at angle θ. The base of the ramp is 30 feet and its upper end is 10 feet from the base, making the length of the ramp $10\sqrt{10}$.



The forces external to the box are the pull of gravity and the force exerted on the box by the plane. This can be expressed as a point diagram. Place one axis labeled H parallel to the inclined plane; the other axis labeled V is perpendicular to this. Let the center of gravity of the box be the origin.



The force of gravity produces a weight of 200 pounds, $|\vec{W}|$, directed toward the center of the earth. Since the box is not moving, the plane must be opposing this force with another 200-pound force, marked $|\vec{P}|$. What is $\vec{P}$? Actually the plane is doing two things to the box: it is lifting it, and it is preventing it from sliding by friction. So there are two forces: one parallel to the V axis and one parallel to the H axis. How much is each force? Before you can determine the magnitude of each force, notice the right triangles in the diagram. One is the inclined plane itself, and the other is the force diagram: $\vec{P}$, $\vec{P}_H$, $\vec{P}_V$. Since the angle of inclination of the plane and the angle between $\vec{P}$ and $\vec{P}_V$ are both θ, the triangles are similar, which implies that the ratios of corresponding sides are equal and the corresponding trigonometric values will be equal.

First, find the sine and cosine of θ for the "ramp" triangle.

$$\sin\theta = \frac{10}{10\sqrt{10}}, \text{ or } \frac{1}{\sqrt{10}} \qquad \cos\theta = \frac{30}{10\sqrt{10}}, \text{ or } \frac{3}{\sqrt{10}}$$

The sine and cosine of θ for the force triangle are the following.

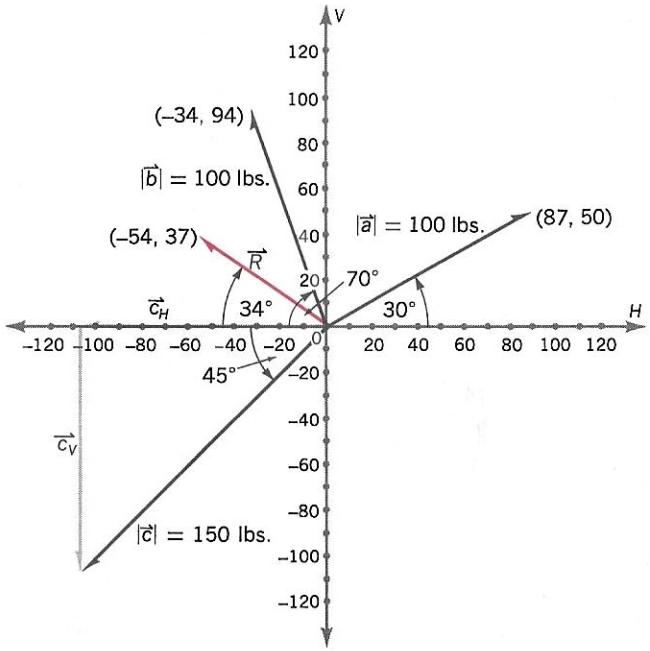
$$\sin\theta = \frac{|\vec{P}_H|}{200} \qquad \cos\theta = \frac{|\vec{P}_V|}{200}$$

Equate the values and solve for the unknown forces.

$$\frac{|\vec{P}_H|}{200} = \frac{1}{\sqrt{10}} \qquad \frac{|\vec{P}_V|}{200} = \frac{3}{\sqrt{10}}$$

$$|\vec{P}_H| = \frac{200}{\sqrt{10}} \approx 63 \text{ pounds} \qquad |\vec{P}_V| = \frac{600}{\sqrt{10}} \approx 190 \text{ pounds}$$

Thus, you have used the physical dimensions of a system to determine the forces acting. Now suppose a series of forces are acting on a point and the magnitude and direction angles are given for the forces. How would you determine the resultant of the forces? You could determine the horizontal and vertical components by using trigonometry, thus determining the endpoints of the vector, and then add by vector addition.



You have been given the coordinates for all the vectors in the figure on the previous page except $\vec{c}$. Determine $|\overrightarrow{c_H}|$ and $|\overrightarrow{c_V}|$ as follows.

$$\sin 45° = \frac{|\overrightarrow{c_V}|}{150} \qquad \cos 45° = \frac{|\overrightarrow{c_H}|}{150}$$

$$|\overrightarrow{c_V}| = \sin 45° \cdot 150 \qquad |\overrightarrow{c_H}| = \cos 45° \cdot 150$$
$$= .7071 \cdot 150 \qquad = .7071 \cdot 150$$
$$\approx 107 \text{ pounds} \qquad \approx 107 \text{ pounds}$$

Thus, the coordinates of the endpoint of $\vec{c}$ are $(-107, -107)$. Be careful to use the correct sign for the coordinates.

Then the resultant, $\vec{R}$, will be

$$\vec{R} = (87 - 34 - 107, 50 + 94 - 107) = (-54, 37).$$

The magnitude of $\vec{R}$, $|\vec{R}|$, is found by the Pythagorean theorem and the angle of inclination is found by trigonometry.

$$|\vec{R}| = \sqrt{(-54)^2 + 37^2} = \sqrt{4285} \approx 65$$
$$\tan \theta = \tfrac{37}{54} \approx .6852$$
$$\theta \approx 34°$$

Thus, $\vec{R}$ is as shown in red in the figure.

Exercises

1. A 400-pound force pointing down to the right makes an angle of 30° with the horizontal. Find the horizontal and vertical components of the force.

2. The components of a force are $|F_H| = 60$ pounds directed to the right and $|F_V| = 100$ pounds directed downward. Determine the magnitude and inclination of the force.

3. Determine the resultant of the force system in Figure 1.

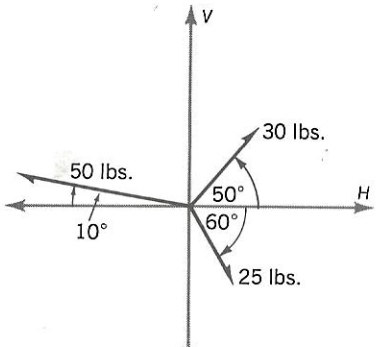


Figure 1

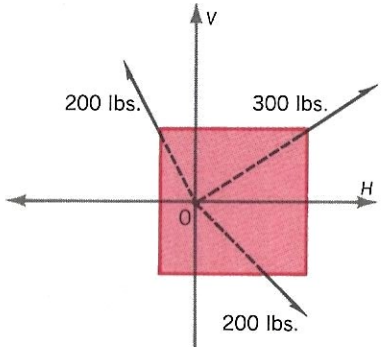


Figure 2

4. In Figure 2 at the bottom of the previous page, forces are applied to a metal plate so that their lines of action all meet in a point, 0. In this case the effect of the force system is equivalent to one with the same forces actually applied at 0. Find the resultant of the force system both in ordered-pair form and in the form of a magnitude and inclination angle.

5. A barrel weighing 500 pounds is on an inclined plane that rises 10 feet from a base 50 feet in length. What force must be applied to the barrel to keep it from rolling down the incline?

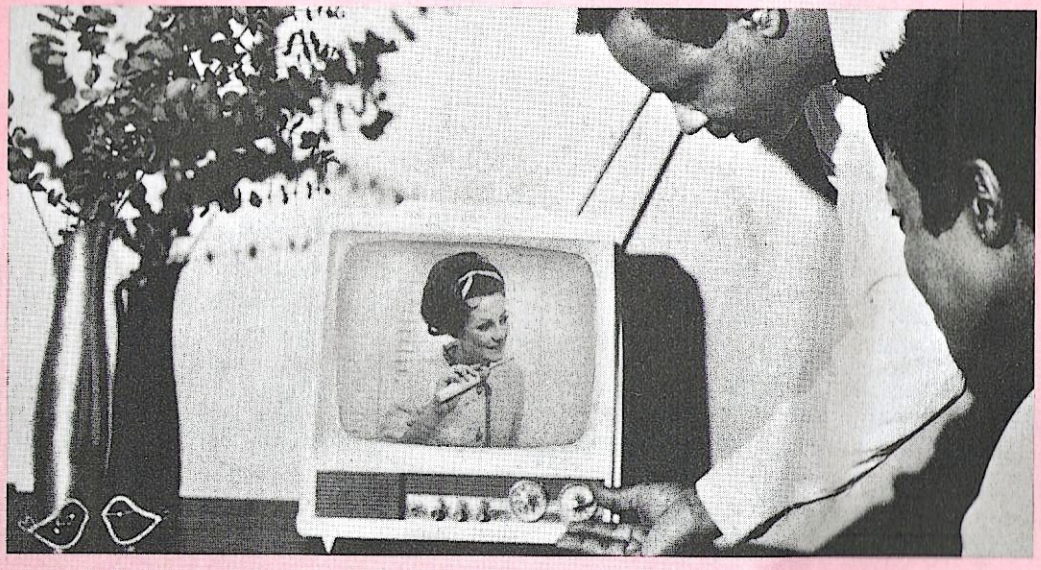
THE GROWTH OF MATHEMATICS

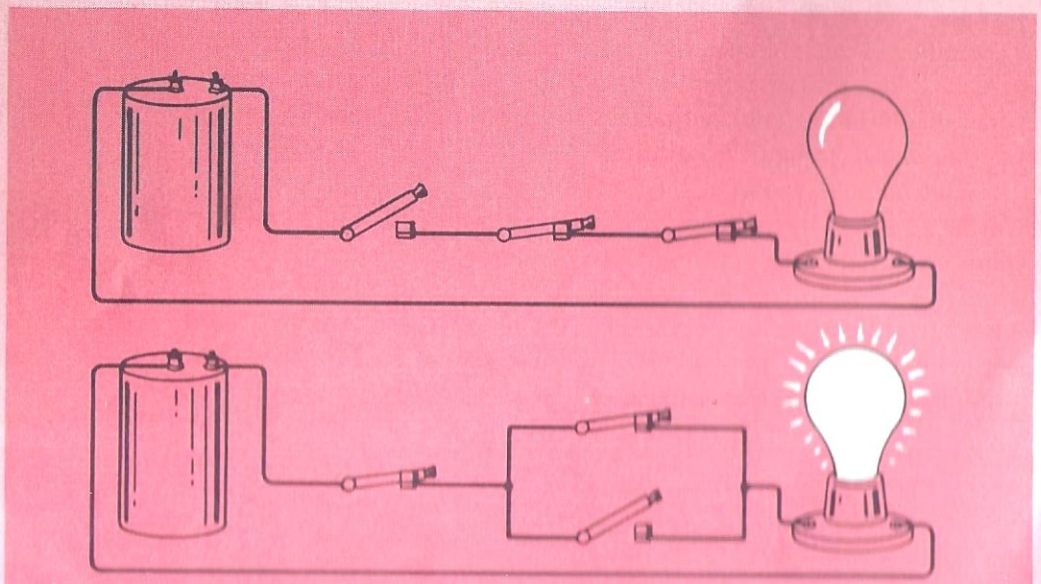
The Cubic Quarrel

Equations involving the third power of the variable and no higher powers are called cubic equations. While mathematicians in India and in the Arabic countries had been able to solve all quadratic equations sometime before A.D. 1000, cubic equations had been solved in only a few special cases by the beginning of the sixteenth century. When the general solution to cubic equations was finally reached, it led to mathematical intrigues up and down the Italian Peninsula.

In 1505 in Bologna, Scipione Ferro (1465–1526) discovered a way to solve some kinds of cubic equations. He told one of his pupils of the solutions, but he did not publish the results. In 1530 Niccolò Tartaglia (1500?–1557) announced that he could solve some cubic equations, so Ferro's pupil challenged him to a mathematical duel. The pupil proposed thirty problems that, as it happened, all were of the kind Tartaglia knew how to solve. Tartaglia proposed thirty problems of another type. His opponent could not solve them, so Tartaglia was declared the winner. Tartaglia could not solve the problems he had proposed either, but, working in Venice, he did manage the general solution of cubic equations in 1541. In the spirit of the times, he announced he had the solution but refused to tell what it was.

Geronimo Cardano (1501–1576), known in English as Jerome Cardan, became interested in the quarrel between Tartaglia and Ferro's pupil. Cardan lured Tartaglia to Milan with false promises, and, on a pledge of strict secrecy, obtained the general solution from him. Then in 1545, Cardan published it as his own. Naturally, another mathematical duel was set up, but it became a near riot between supporters of Cardan and those of Tartaglia. It is believed that Tartaglia won, but the general solution is known as "Cardan's solution."





CHAPTER 12
LOGIC AND PROOF

Much of the reading, writing, and speaking done by a person involves a chain of reasons and a conclusion. The science concerned with the principles of correct reasoning is logic.

● A lawyer presenting a case before a court depends mainly on logic. Point by point, he tries to convince the judge and jury that his conclusion is the correct conclusion. Consider the following statements.

Object A was stolen from Mr. X.
Object A was found in the possession of Mr. Z.
If Mr. Z has object A, then he must have stolen it from Mr. X.

Is this reasoning enough to convince you that Mr. Z is the thief? What if someone gave him object A, or he found object A on the street? More arguments are needed to complete the chain.

● The electric circuits pictured at the bottom of the page have switches that regulate the current going from the battery to the light bulb. If the switch is closed, the current can pass through the wire. If it is open, the current cannot pass through that wire. One switch in the top circuit is open and one switch in the bottom circuit is open. Why is the bulb lit in the bottom circuit but not in the top circuit?

You will see in this chapter how the switches in the circuit can be used to represent statements and the bulb's being on or off will tell you if the logical conclusion is true or false.

● Advertising campaigns sometimes rely on a person's drawing more than the obvious conclusion from an ad. What do many toothpaste advertisers hope you will think after viewing their ads? Probably, that if you use their toothpaste, in addition to having clean teeth and fewer cavities, you will have a lovely smile and a sparkling personality. But there is nothing in the evidence to prove this. In this chapter you will see how logic is applied to reasoning and proofs.

12–1 Statements

The primary components of logic are sentences that can be classified as true or false. Such sentences are called **statements.** Listed below are six sentences, three of which, **d–f,** are statements.

a. x is an even number.
b. He is in my class.
c. $6 + x = 10$
d. 8 is an even number.
e. This book is an algebra book.
f. $6 + 5 = 11$

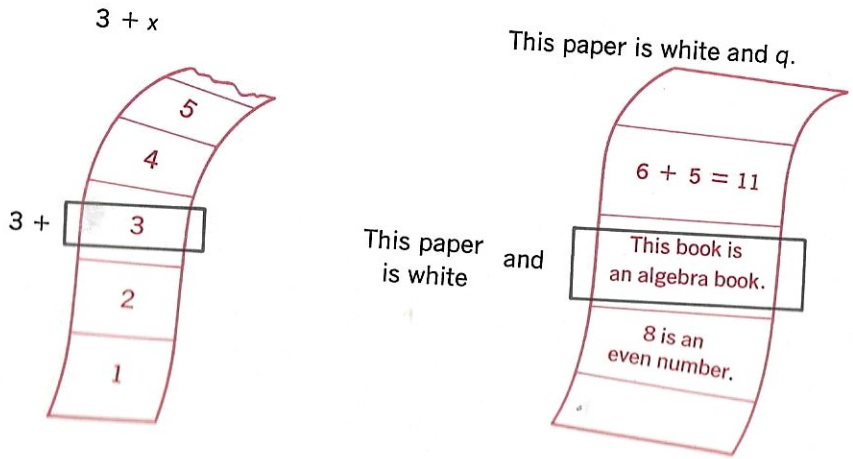
Why can sentences **a–c** not be classified as true or false?

Definition A statement is a sentence that makes an assertion that is either true or false.

In mathematics and logic, you only need to consider sentences that are statements and open sentences that become statements when the variable is replaced by an element from its replacement set. Thus, you will not work with such nonstatements as "Close the door" or "Does $6 + x$ equal 10?"

An open sentence such as $x + 2x = 3x$ becomes a statement if the variable is quantified. Thus, $\forall x \in R \quad x + 2x = 3x$ is a statement.

In logic, letters are used as variables as they are in algebra. But instead of representing a number, the letter represents a simple statement. Lower-case letters, such as p, q, r, and s, will be used in general statement forms, whereas capital letters represent particular statements. The following diagram compares the use of x as a variable for numbers and q as a variable for statements.



Exercises

Identify the sentences that are statements.

1. Red is a color.
2. Frank is 14 years old.
3. She is a cheerleader.
4. $2 + 9 = 11$
5. $\forall a \in R \quad a + 0 = a$
6. There are 366 days in a leap year.
7. $x + y = xy$
8. Every month has 30 days.
9. All birds can fly.
10. Q is a false statement.

Identify the sentences. Which are also statements?

11. In the universe.
12. $\exists y \in R \quad y + 2 = 5$
13. $7 \div 8$
14. c is an element of $\{a, b, c\}$.
15. The set of real numbers is a subset of set X.
16. There is a one-to-one correspondence between points on a line and the real numbers.
17. Are all elephants.
18. If x is 1, 2, or 3, $x + 1 < 5$ is true.
19. $\frac{2}{3} = \frac{3}{9}$
20. Of all the countries he visited.

12–2 Conjunctions and Disjunctions

The statements in Section 12–1 can be classified as **simple statements.** Simple statements joined by a connective, such as *and* and *or*, form **compound statements.** A compound statement with the connective *and* is a **conjunction.** *This paper is white* is joined by *and* with three statements to form three conjunctions.

This paper is white *and* q.
This paper is white *and* 8 is an even number.
This paper is white *and* This book is an algebra book.
This paper is white *and* $6 + 5 = 11$

If the symbol $\wedge$ is used for *and* and p is used to represent *This paper is white*, the conjunctions can be symbolized as $p \wedge q$. The conjunction $p \wedge q$ is true only when p is true and also q is true. Are the three conjunctions above true?

Study the tables below. At the left is a table of simple statements for p and q in all the possible combinations of true and false statements. At the right is a table with just the **truth values**, that is, T for true and F for false, of the statements and the conjunctions. It is called a **truth table.** Why is the conjunction true only for the first case?

p	q
Snow is white.	Grass is green.
Snow is white.	Grass is purple.
Snow is blue.	Grass is green.
Snow is blue.	Grass is purple.

		Conjunction
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The compound statement formed by two simple statements joined by *or* is a **disjunction.** The symbol $\vee$ is used for *or*. The *or* in logic means the same as *either . . . or . . . or both* in ordinary language. Hence, $p \vee q$ is true when p is true, when q is true, or when both p and q are true.

Use the same simple statements as above to explain why the disjunction is false only in the last case shown in the truth table.

Notice that the T's and F's in the left part of the table always appear in the same pattern, including all the possible combinations of T and F.

		Disjunction
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

You have used both conjunctions and disjunctions when you solved compound sentences such as $x < 5$ *and* $x > 3$ and $x < 5$ *or* $x > 3$. To solve the compound sentences, you found the numbers that make true compound statements. The solution set for the compound sentence can be denoted using *intersection* or *union* of sets.

EXAMPLE 1. Find the solution set for $(x < 5) \wedge (x > 3)$.

Compound sentence	Some compound statements
$(x < 5) \wedge (x > 3)$	$(2 < 5) \wedge (2 > 3)$
	$(3 < 5) \wedge (3 > 3)$
	$(3\frac{1}{2} < 5) \wedge (3\frac{1}{2} > 3)$
	$(4 < 5) \wedge (4 > 3)$
	$(4\frac{7}{8} < 5) \wedge (4\frac{7}{8} > 3)$
	$(5 < 5) \wedge (5 > 3)$
	$(6 < 5) \wedge (6 > 3)$

Solution set: $\{x : x < 5\} \cap \{x : x > 3\}$

Which replacements for x make $x < 5$ true? Which replacements for x make $x > 3$ true? The numbers shown in red are in the solution set of the compound sentence; that is, they are in the intersection of the two sets that make each of the simple sentences true. The entire solution set is shown in set-builder notation and by the following number-line graph.

Use the same process to explain the following example.

EXAMPLE 2. Find the solution set for $(x < 5) \vee (x > 3)$.

Solution set: $\{x : x < 5\} \cup \{x : x > 3\}$

Thus, the numbers that make true conjunctions can be denoted by using intersection of sets, and the numbers that make true disjunctions can be denoted by using union of sets.

Checkpoint

1. Explain the symbols $\wedge$ and $\vee$.
2. What is a compound statement?
3. What is a conjunction? a disjunction?

Exercises

A — Form the conjunction of each pair of statements given below. Indicate whether the statement formed is *True* or *False*.

1. $2 \times 5 = 10$; $3 \times 8 = 36$
2. $6 + 8 = 14$; $3 + 2 = 5$
3. $8 \times 7 = 55$; $5 + 4 = 8$
4. $17 = 3 \times 5 + 2$; $32 = 6 + 2 \times 4$
5. $18 = 4 \times 4 + 2$; $16 = 3 \times 5 + 1$

— Form the disjunction of each pair of statements given below. Indicate whether the statement formed is *True* or *False*.

6. $5 > 2$; $-3 < -4$
7. $3 - 2 = 0$; $5 + \frac{1}{5} = 1$
8. $3 \neq 2$; $-6 > -5$

9. $5(1 + 8) = 5 + 40$; $8 \div 2 = 2 \div \frac{1}{2}$

10. $-2 \div 0 = 0$; $5 \times 0 = 1$

Find and graph the solution set for each of the following compound sentences. The replacement set is R.

11. $x > 6$ *and* $x < 10$

12. $x + 4 = -1$ *or* $x + 2 < 5$

13. $4(x + 1) = 12$ *and* $x = 2$

14. $2x + 3 < 4$ *or* $x + 5 = -4$

15. $3x + 9 < 7$ *and* $2x + 3 > -2$

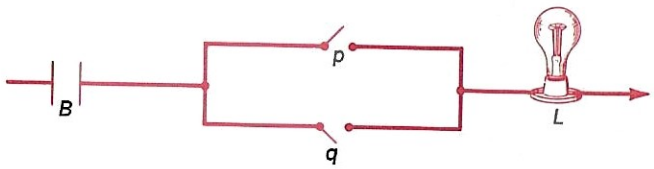
Make truth tables for the following compound statements.

16. $p \vee q$ **17.** $p \wedge q$ **18.** $p \wedge r$

19. $q \wedge r$ **20.** $q \vee r$ **21.** $p \vee r$

22. If p and q are true and r is false, tell whether the compound statements in Exercises 16–21 are *True* or *False*.

B **23.** The truth value of a disjunction can be represented by an electrical circuit wired in parallel as shown in the diagram. B is the battery; p and q are switches; L is the light.



An open switch at both p and q indicates that current cannot flow from B to L. If either p or q or both are closed, the current can flow from B to L and the light will be on.

Given the following positions for switches p and q, indicate whether the light will be *On* or *Off*.

a. p is open; q is open. **b.** p is open; q is closed.

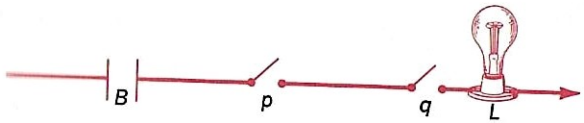
c. p is closed; q is open. **d.** p is closed; q is closed.

24. Summarize your results from Exercise 23 by completing the table at the right. T means the switch is closed, or the light is on. F means the switch is open, or the light is off.

Compare your results with the truth table for a disjunction.

Switch		Light
p	q	
T	T	
T	F	
F	T	
F	F	

25. The truth value of a conjunction can be represented by an electric circuit wired in series as shown in the diagram.



Using this circuit, write *On* or *Off* for **a–d** in Exercise 23. Then summarize your results in a table as in Exercise 24. Compare the table with the truth table for a conjunction.

12–3 Conditionals

A teacher made the following statement to illustrate what is called a **conditional statement.**

If your test score is A, *then* you have no homework.
(P) (Q)

Definition A conditional is a compound statement made up of two statements joined in *If . . . then . . .* form.

The conditional *If* p, *then* q is symbolized as $p \rightarrow q$, which is read "p implies q."

To determine the truth value of $p \rightarrow q$, look at the tables below.

P: Your test score is A.
Q: You have no homework.

Interpret $P \rightarrow Q$ as "The teacher kept his promise."

		Did the teacher keep his promise?
Your test score is A.	You have no homework.	Yes
Your test score is A.	You have homework.	No
Your test score is not A.	You have no homework.	Yes
Your test score is not A.	You have homework.	Yes

The first, second, and fourth results are easy to see. What about the third? The reason for *Yes* in the third is that the teacher only

promised what would happen if your test score *was* A. Logically, you cannot conclude that a false promise was made if your test score was not *A*. So the answer is *Yes*.

The results are summarized in the following truth table.

		Conditional
p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

A conditional is false only when p is true and q is false. Use the conditional "If it is raining, then I use my umbrella" to verify the truth table for conditionals.

In ordinary language, a conditional statement would be interpreted as meaning that p *causes* q to happen. This is not the case in logic. Statements p and q are independent statements with no causal connection intended between them. They can be treated as separate parts.

Suppose p and q are switched. Then the statement would be $q \to p$.

Definition The converse of a conditional is the statement formed by interchanging p and q in $p \to q$. The converse of $p \to q$ is $q \to p$.

EXAMPLE 1. Is the converse of "If a number is divisible by 4, then the number is an even number" true?

P: A number is divisible by 4.
Q: The number is an even number.
$Q \to P$: If a number is an even number, then the number is divisible by 4.

The converse, $Q \to P$, is false because not all even numbers are divisible by 4.

EXAMPLE 2. What is the converse of "If today is July 4, then today is Independence Day"? Is it true or false?

P: Today is July 4.
Q: Today is Independence Day.
$Q \to P$: If today is Independence Day, then today is July 4.

The converse is true.

If a conditional and its converse are both true, they can be combined into one true statement called a **biconditional.** The connective is *if and only if*.

Conditional: If today is July 4, then today is Independence Day.
Converse: If today is Independence Day, then today is July 4.
Biconditional: Today is July 4 if and only if today is Independence Day.

The symbol for *if and only if* is $\leftrightarrow$. The conjunction can be expressed symbolically as follows.

$$(p \to q) \wedge (q \to p)$$
$$p \leftrightarrow q$$

The truth table below summarizes the results for the conditional, its converse, and the biconditional. Cover those three columns and see if you can give the correct truth values before looking at the results.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

EXAMPLE 3. Form the conditional, converse, and biconditional for the statements below. Identify each as true or false.

P: x is an even number.
Q: x is divisible by two.
$P \to Q$: If x is an even number, then x is divisible by two. T
$Q \to P$: If x is divisible by two, then x is an even number. T
$P \leftrightarrow Q$: x is an even number if and only if x is divisible by two. T

Checkpoint

1. What is a conditional? the converse of a conditional?
2. What is a biconditional? When is a biconditional true?

Now try these

Is each implication true or false?

1. If 2×3 equals 6, then $5 + 2$ equals 7.
2. If a is an integer, then a is a rational number.

3. Is the converse of each statement in Exercises 1 and 2 true or false?

4. Form the biconditional for the statement in Exercise 2. Is it true or false?

Answers: **1.** T **2.** T **3.** T; F **4.** a is an integer if and only if a is a rational number; F.

Exercises

A — You are given the following statements.

P: E is a vowel.
Q: There are six days in a week.
R: The product of 2 and 1 is 21.
S: All ridable bicycles have wheels.

Write a conditional statement in *If . . . then . . .* form for each of the following symbolic statements. Indicate whether each is *True* or *False*.

1. $P \to Q$ **2.** $Q \to S$ **3.** $S \to P$
4. $Q \to R$ **5.** $R \to P$ **6.** $S \to R$

7. Write out the converse for each of the statements in Exercises 1–6. Indicate whether each is *True* or *False*.

— Which pairs of statements combine to form a biconditional that is true?

8. P: You travel to Mexico.
Q: You drive a car to Mexico.

9. P: A triangle is equilateral.
Q: A triangle is equiangular.

10. P: x is a prime number.
Q: x is not divisible by 5.

11. P: x is an odd integer.
Q: x is an integer not divisible by 2.

12. P: This flower is a rose.
Q: This flower is red.

13. P: 9 is a factor of n.
Q: 3 is a factor of n.

B — Using the four statements at the beginning of this exercise set indicate whether the following statements are *True* or *False*.

14. $(P \wedge Q) \to R$ **15.** $(P \vee R) \to S$
16. $(R \wedge P) \to Q$ **17.** $S \to (R \vee Q)$
18. $P \to (Q \vee S)$ **19.** $(R \wedge S) \to P$

12–4 Negation

The **negation** of a simple statement is formed by inserting *It is false that . . .* at the beginning of the statement. This is the same as negating the verb, as shown in the following examples.

Statement	*Negation*
P: 5 is an odd number.	*It is false that* 5 is an odd number. 5 is *not* an odd number.
Q: $\sqrt{2}$ is a rational number.	*It is false that* $\sqrt{2}$ is a rational number. $\sqrt{2}$ is *not* a rational number.

Statement P is true. Is its negation true? Is the negation of a true statement ever true?

Statement Q is false. But its negation is true. Is the negation of a false statement always true?

The negation of p is symbolized as $\sim p$ and is read "not p."

If both parts of a conditional are negated, another type of statement called the **inverse** is formed. The inverse of $p \to q$ is $\sim p \to \sim q$.

EXAMPLE. Write the inverse of the following statement: If a number is divisible by 6, then the number is an even number.

P: A number is divisible by 6.
Q: The number is an even number.
$\sim P \to \sim Q$: If a number is not divisible by 6, then the number is not an even number.

In this case, the inverse is not true. Look at the truth table below and explain why.

p	q	$\sim p$	$\sim q$	$\sim p \to \sim q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Negations of disjunctions and conjunctions are a little more complicated.

The disjunction "Snow is white *or* ice is cold" is true if *either* of its statement parts is true. The negation is "It is false that snow is white *or* ice is cold"; that is, it is false that one or the other or both are

true. Thus, both parts are false, and the negation can be written "It is false that snow is white *and* it is false that ice is cold."

$$\sim(p \vee q) \quad \text{is} \quad \sim p \wedge \sim q$$

The negation of a disjunction is a conjunction formed from the negation of both parts of the disjunction.

The conjunction "Snow is white *and* ice is cold" is true only when *both* parts are true. The negation is "It is false that snow is white *and* ice is cold." Since the conjunction is false if *one or the other or both parts* are false, the negation can be written "It is false that snow is white *or* it is false that ice is cold."

$$\sim(p \wedge q) \quad \text{is} \quad \sim p \vee \sim q$$

The negation of a conjunction is a disjunction formed from the negation of both parts of the conjunction.

The reasoning above uses practical examples to show that $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are **logically equivalent** statements. Another way is to make a truth table for each statement. If the truth values for the statements are exactly alike, then the *statements are logically equivalent*. If p is logically equivalent to q, you can write $p \leftrightarrow q$. Note that the symbol is the same as that for the biconditional, as the meaning is the same.

Cover all but the first two columns of the truth tables below and see if you can correctly complete them.

p	q	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Compare the last columns in each table. Are they exactly alike? Then $[\sim(p \vee q)] \leftrightarrow [\sim p \wedge \sim q]$ is true.

Exercises

A — Form the negation of each of the following statements and determine if each negation is true or false.

1. A square has four sides that are congruent.
2. An integer is a rational number.
3. A rational number is an irrational number.
4. A function is a set of ordered pairs.
5. An irrational number is a real number.
6. The number of natural numbers exceeds the number of even natural numbers.
7. $5 \times 3 - 4 \div 2 + 1 = 14$
8. Zero is a positive integer.
9. $\log_{10} 0 = 0$
10. Two perpendicular lines intersect.

— First determine the truth value of each conditional. Then write its inverse and determine its truth value.

11. If a number is divisible by 9, then it is a multiple of 3.
12. If a triangle is equilateral, then three sides of the triangle are the same length.
13. If $3 + 2 = 5$, then $2(3 + 2) = 2(5)$.

B — Form the negation of the following conjunctions and disjunctions.

14. Math is fun *and* logic is easy.
15. Winter is cold *or* summer is warm.
16. $x > 2$ *and* $x < 5$
17. $3 \times 5 = 4$ *or* $6 - 2 = 4$
18. $23 = 3 + 4 \times 6$ *and* $13 = 4 \times 2 + 5$

— Replace each $\underline{?}$ by the symbol that completes each statement so that it is true.

19. $\sim(p \vee r) \leftrightarrow \sim p \ \underline{?} \ \sim r$
20. $\sim(r \wedge s) \leftrightarrow \sim r \ \underline{?} \ \sim s$
21. $\sim p \wedge \sim q \leftrightarrow \underline{?} \vee \underline{?}$
22. $\sim s \vee \sim t \leftrightarrow \underline{?} \wedge \underline{?}$
23. $\underline{?}(q \wedge r) \ \underline{?} \ \sim q \vee \sim r$
24. Make a truth table for $\sim(p \wedge q)$ and for $\sim p \vee \sim q$. How do the truth values for the statements compare?

25. Make a truth table to show that the inverse of $p \to q$ may be false when $p \to q$ is true.

26. The statement $\sim q \to \sim p$ is called the **contrapositive** of $p \to q$. As you can see, it is formed by both negating and switching the two parts of the conditional. Make a truth table to show that $p \to q$ is logically equivalent to $\sim q \to \sim p$.

27. Make truth tables to show that the converse and inverse of a given conditional are logically equivalent.

28. Write the converse, inverse, and contrapositive of the following conditionals.

a. $\sim p \to q$ **b.** $p \to \sim q$

c. If the temperature of water is lower than 32° F, the water will freeze.

29. The following truth table involves three statements, so eight rows of T's and F's are needed. The pattern in the three columns is the same as it was for the two-statement table: the first column has the first half T's and the second half F's; the second column has one fourth of the column T's, one fourth F's, and so forth; the third column has one eighth of the column T's, one eighth F's, which here means alternating T and F.

Replace each ? by T or F to correctly complete the chart.

p	q	r	$p \wedge q$	$(p \wedge q) \to r$
T	T	T	T	?
T	T	F	?	?
T	F	T	?	T
T	F	F	F	?
F	T	T	?	?
F	T	F	?	T
F	F	T	F	?
F	F	F	?	T

12–5 Tautologies

Sometimes statements are combined and the resulting statement is true, regardless of the truth values of the original statements. A statement form that has this property is called a **tautology.**

The example on the following page illustrates the idea of a tautology.

EXAMPLE Show that $p \vee \sim p$ is a tautology.

Construct a truth table. Under p, write the possible values: T and F. Then fill in the other columns with the truth values for negation and disjunction.

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

The third column indicates that whether the value of p is true or false, the disjunction is always true. Thus, it is a tautology.

The statement $p \vee \sim p$ is known as the **Law of the Excluded Middle.** An example would be "Today is Monday *or* today is not Monday." You can see that this is always true.

Exercises

A Use truth tables to show that the following statements are tautologies.

1. $\sim(\sim p) \leftrightarrow p$

2. $(p \leftrightarrow q) \leftrightarrow [(p \to q) \wedge (q \to p)]$

3. $(p \wedge q) \to p$

4. $(p \wedge \sim q) \vee \sim(p \wedge \sim q)$

Use truth tables to show whether or not the following are tautologies.

5. $\sim p \vee p$

6. $q \vee q$

7. $p \wedge p$

8. $\sim(\sim q) \vee \sim q$

9. $(p \wedge q) \wedge (\sim p \vee q)$

10. $\sim(\sim p) \wedge \sim p$

B **11.** At the right is the truth table for $p \wedge \sim p$. The truth value of the conjunction is always false. A statement that is always false is a **contradiction.** Construct a truth table to show that $\sim(p \vee \sim p)$ is a contradiction.

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Construct a truth table to show whether each of the following statements is a tautology, a contradiction, or neither.

12. $p \vee (q \vee r) \leftrightarrow (p \vee q) \vee r$

13. $[(p \to q) \wedge (q \to r)] \to (p \to r)$

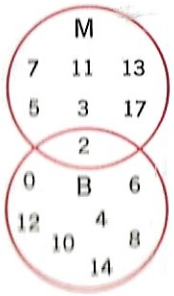
14. $\sim[(p \vee q) \to (q \vee p)]$

15. It is false that if John is 21 and all men 21 or over can vote, then John can vote.

16. $2 + 7 = 9$ *or* $10 \div 2 = 6$ implies that $10 \div 2 = 6$ *or* $2 + 7 = 9$.

12–6 Quantifiers

Sets M and B are illustrated in the diagram. The sets are described below by sentences using **quantifiers.**



a. *All* numbers in M are prime.

b. *Each* number in M is prime.

c. *Any* number in M is prime.

d. *Some* number in B is prime.

e. *There exists* a number in B that is prime.

f. *At least one* number in B is prime.

Sentences **a–c** use *all*, *each*, and *any* to denote exactly the same meaning, and they indicate the same quantification, or the same number of members. This quantification is shown, as you know, by the symbol $\forall$. Thus, the following sentence is true.

$$\forall x \in \mathrm{M} \quad x \text{ is prime.}$$

In sentences **d–f**, the quantifiers are *some*, *there exists*, and *at least one*. In ordinary usage, *some* usually means "at least two but not all." But in logic *some* means the same as *there exists* and *at least one;* that is, it could denote one or all, but at least one. The quantification is shown by the symbol $\exists$. The following sentence is true.

$$\exists x \in \mathrm{B} \quad x \text{ is prime.}$$

Negation and Quantifiers

Why is the sentence "All numbers in M are odd" false? It takes only one counterexample to make the *all* statement false, and in this case, 2 makes it false. Hence, "It is false that all numbers in M are odd" is equivalent to "At least one number in M is not odd."

The negation for a quantified statement using *all*, *each*, or *any* is a quantified statement using *some*, *there exists*, or *at least one* with the verb negated.

Suppose you had to decide if the statement "There exists a multiple of 20 in set B" were true or false. You would examine all the members in set B and conclude that the statement is false. You could then say any of the following were true.

All numbers in B *are not* multiples of 20.
No number in B *is* a multiple of 20.
Each number in B *is not* a multiple of 20.

The negation for a quantified statement using *some*, *there exists*, or *at least one* is a quantified statement using *all*, *each*, or *any* with the verb negated.

Note that *All x is not . . .* could also be worded *No x is . . .*

Exercises

A — Write T or F for each statement. Then form the negation of each statement and label it T or F.

1. All numbers are negative.

2. At least one rational number is not an integer.

3. Some triangles have three sides congruent.

4. Each graph represents a relation.

5. All terminating decimals are rational numbers.

6. Some straight lines represent functions.

7. All numbers that cannot be expressed as the ratio of two integers are irrational numbers.

8. Some numbers are not whole numbers.

9. Any directed line segment is a vector.

10. Some numbers make $x^2 - 5x + 6 = 0$ true.

Example. $\exists x > 2 \quad x + x = x^2$

The statement is false.

The negation is $\sim(\exists x > 2 \quad x + x = x^2)$

or $\forall x > 2 \quad \sim(x + x = x^2)$

or $\forall x > 2 \quad x + x \neq x^2$.

The negation is true.

11. $\forall x \quad x + 2x = 3x$

12. $\exists x \quad x + 7 = 19\frac{1}{2}$

13. $\exists x \quad x + x > x + 9$

14. $\forall x \quad |x| \geq x$

15. $\forall x \quad |x| \geq 0$

16. $\exists x \quad x^2 + 5x + 6 = 0$

17. $\exists x \in \mathrm{R} \quad x^2 = 2$

18. $\exists x \quad 9x - 5x = 4x$

— Replace each ? below by ∀x∈R or ∃x∈R to make a statement true.

19. ? $x + x = 2x$

20. ? $x^2 + 4x - 12 = 0$

21. ? $\frac{4x}{x} = 4$

22. ? $x \cdot x = x^2$

23. ? $(x + 6)(x - 5) = x^2 + x - 30 = 0$

B — Indicate whether each of the following is true, T, or false, F.

24. $\forall x \in R \quad \forall y \in R \quad x + y = y + x$

25. $\exists x \in R \quad \frac{x}{0} = 1$

26. $\forall x \in R,\ x \neq 0 \quad x \cdot \frac{1}{x} = 1$

27. $\exists x \in R \quad 2x - x = 2$

28. $\forall x \in R \quad \forall y \in R \quad \forall z \in R \quad (x - y) - z = x - (y - z)$

29. $\exists x \in R \quad |x - 3| = -1$

30. $\forall x \in R \quad x^2 - x + 11$ is a prime number.

31. $\exists x \in R \quad 2x + x = 2x$

32. $\forall x \in R \quad (3x - x = 2x) \wedge (4x = 20)$

33. $\forall x \in R \quad (3x > 2x) \wedge (|x| > 0)$

34. $\exists x \in R \quad \left(\frac{x}{6} = 0\right) \wedge (6 \cdot x = 0)$

35. $\exists x \in R \quad \exists y \in R \quad (x + y = 10) \wedge (x - y = 10)$

36. $\exists x \in R \quad \exists y \in R \quad (5x + y = 8) \wedge (5y = 40 - 25x)$

C **37.** If $[\sim\forall x \quad \sim(x + 1 = 1 + x)] \leftrightarrow [\exists x \quad x + 1 = 1 + x]$ can be abbreviated to $\sim\forall\sim \leftrightarrow \exists$, abbreviate $[\forall x \quad x \cdot 1 = 1 \cdot x] \leftrightarrow [\sim\exists x \quad \sim(x \cdot 1 = 1 \cdot x)]$ in the same way.

38. Make abbreviated forms for the two general statements regarding negation and quantifiers in this section.

12–7 Definitions and Undefined Terms

Proof in mathematics begins with certain undefined elements or terms, postulates (assumed statements) about the elements, and definitions of new elements or terms. A theorem is established using the principles of logic with postulates and definitions. Subsequent theorems may use previously proved theorems.

In this section, you will take a more careful look at the way definitions are formulated and at the role of undefined terms.

Four statements are listed below. One is a good definition of *square*. Which is it?

a. A square is a four-sided figure.

b. A square is a figure that has four congruent sides. (Note: *Congruent* means "having the same measure.")

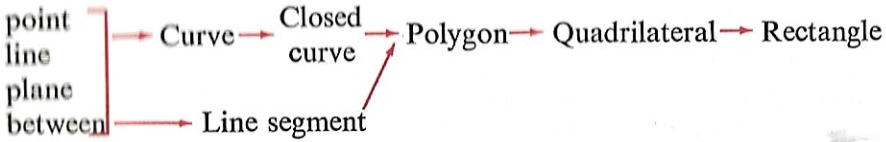
c. A square is a rectangle having two adjacent sides congruent.

d. A square is a figure that has four congruent sides, four congruent angles, and two congruent diagonals.

Most people would agree that **c** is the best definition. The reason is most easily seen by looking at the following chart. The left column gives characteristics of a good definition, the right column discusses **c**.

1. The technical words used to define the term should have been defined previously or they should be in the set of undefined terms.	Technical words previously defined would include *rectangle*, *adjacent sides*, and *congruent*.
2. The term being defined should be placed in the smallest previously defined set to which it belongs.	Why was *rectangle* chosen as the smallest previously defined set instead of *quadrilateral* or *parallelogram*?
3. Just enough characteristics are given to distinguish it from other elements in the same set.	Does *two adjacent sides congruent* distinguish the elements called squares from all the other rectangles? The characteristic that all sides are congruent was not included because this can be proved as a theorem.
4. The statement is reversible; that is, separating the statement into two statements p and q produces $p \leftrightarrow q$.	P: A figure is a square. Q: A rectangle has two adjacent sides congruent and one right angle. $P \rightarrow Q$: If a figure is a square, it is a rectangle with two adjacent sides congruent and one right angle. $Q \rightarrow P$: If a rectangle has two adjacent sides congruent and one right angle, the figure (or rectangle) is a square. Since $P \rightarrow Q$ and $Q \rightarrow P$ are true, $P \leftrightarrow Q$ is true.

You can trace the technical words used in any definition back to undefined terms. For example, *rectangle* can be traced back to the undefined terms *point*, *line*, *plane*, and *between* as shown in the diagram below.



Begin with the undefined terms and define each subsequent term, using only the undefined terms or your previous definitions.

Exercises

A — The following statements define terms that you have used in algebra. Each statement lacks at least one of the properties of a good definition. Which properties are lacking? Formulate an acceptable definition.

1. A *function* pairs members of the domain with members of the range.

2. A *rational number* is a number of the form $\frac{a}{b}$ where $b \neq 0$.

3. A *variable* is a symbol.

4. A nonterminating, nonrepeating decimal is a *real number*.

5. An *equation* is a statement like $4x + 1 = 6$.

— State each definition in an acceptable *If . . . then . . .* form. Then write its converse. Are both statements you wrote true?

6. An *inequality* is a statement that two numbers or algebraic expressions are not equal.

7. An *exponential function* is a function defined by $y = b^x$, where $b \neq 1$ and $b > 0$.

8. The *cosine* of an acute angle is the ratio

$$\frac{|\text{ side adjacent to acute angle }|}{|\text{ hypotenuse }|}.$$

9. An *irrational number* is a number that cannot be expressed as the ratio of two integers.

10. The *square root* of a number is one of its two equal factors.

B — Justify each of the following statements with a definition you learned earlier in the course.

11. The number $0.14\overline{14}$ is a rational number.

12. $\sqrt{2}$ is an irrational number.

13. The sine of angle A in Figure 1 is $\frac{a}{c}$.

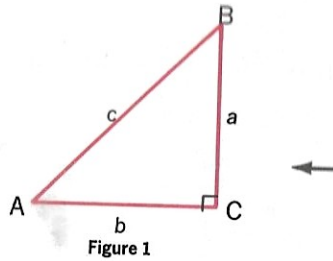


Figure 1

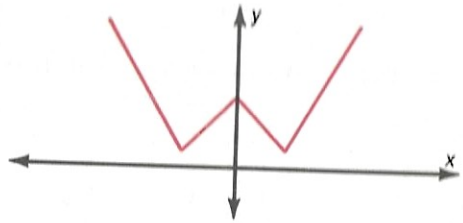


Figure 2

14. The graph in Figure 2 defines a function.

15. $2x + 4 < 6$ is an inequality.

12–8 Rules of Inference

Reasoning from a series of statements, called **premises,** to a final statement, called a **conclusion,** is known in logic as **inference.** The inference is valid if true premises lead to a true conclusion. There are various patterns of inference that are always valid.

Let P and Q denote the following statements.

P: The quadratic equation $x^2 + x - 12 = 0$ has a perfect-square discriminant.

Q: The quadratic equation $x^2 + x - 12 = 0$ has rational roots.

Is $P \rightarrow Q$ a true conditional? Is P a true statement? From your work in logic and your earlier work with quadratic equations, you should conclude that both $P \rightarrow Q$ and P are true. Now without finding the roots of $x^2 + x - 12 = 0$, you can conclude that Q is true; that is, the equation has rational roots.

The truth of this conclusion is determined by **modus ponens,** or the rule of detachment.

Modus ponens If $p \rightarrow q$ is a true statement and if p is a true statement, then q is a true statement.

Symbolically, *modus ponens* may be expressed as follows.

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

You can show that *modus ponens* is valid by a truth table.

p	q	$p \to q$	$(p \to q) \wedge p$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

Hence, $(p \to q) \wedge p$ is true *only* when q is also true.

In an earlier exercise, you proved that $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology. Then if $(p \to q) \wedge (q \to r)$ is true, $p \to r$ is true by **transitivity.**

Transitivity If $p \to q$ is a true statement and $q \to r$ is a true statement, then $p \to r$ is a true statement.

How is the rule of logic like the transitive property for equality? Symbolically, transitivity is expressed as follows.

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

EXAMPLE. Draw a valid conclusion from the following statements, P, Q, and R.

P: The weather is good.
Q: I will go to the football game tomorrow.
R: I will spend one dollar tomorrow.

$P \to Q$: If the weather is good, then I will go to the football game tomorrow.

$Q \to R$: If I go to the football game tomorrow, then I will spend one dollar tomorrow.

$P \to R$: If the weather is good, then I will spend one dollar tomorrow.

If you accept the conditionals $P \to Q$ and $Q \to R$ as true, then by transitivity, you can accept $P \to R$ as true.

Consider the statement "For every real number x, $x + 2x = 3x$ is true," or

$$\forall x \in \mathrm{R} \quad x + 2x = 3x.$$

Since 6 is a real number, you can replace x by 6 and obtain the true statement

$$6 + 2(6) = 3(6).$$

The right to exchange the general statement $x + 2x = 3x$ for the particular instance $6 + 2(6) = 3(6)$ is due to the **rule of substitution for variables.**

Rule of Substitution for Variables If a statement p is true for all permissible replacements in a given set, then it is true for a particular replacement in the set.

In other words, if p is true for every x, then by the rule of substitution, you can replace x by any particular permissible value for x and obtain a true statement.

This means then that the rule does not apply to a variable x in a statement such as

$$\exists x \in \mathrm{R} \quad x + 3 = 7.$$

The quantifier $\exists$ limits the variable to *some* x. If you replaced x by 5, you would get the statement $5 + 3 = 7$, and of course this is not true.

Checkpoint

1. What is meant by *modus ponens*?
2. What is meant by transitivity?
3. What is meant by the rule of substitution for variables?

Exercises

A — Use truth tables to determine whether each of the conclusions is true or false. To be true, the conclusion must be true for all cases in which the given statement is true.

1. *Given:* $\begin{cases} p \to q \\ \sim q \to r \end{cases}$ *Conclusion:* $\{p \to r$

2. *Given:* $\begin{cases} r \to s \\ t \to \sim s \end{cases}$ *Conclusion:* $\{r \to s$

— Read the following statements. Label the statements in each as p and q, and then show that each argument uses *modus ponens*.

3. If a triangle is isosceles, then it has two congruent sides. It is known that triangle ABC is isosceles. Therefore, triangle ABC has two congruent sides.

4. If a, b, and c are positive integers and $a^2 + b^2 = c^2$ is true, then (a, b, c) is called a *Pythagorean triple*. The integers 5, 12, and 13 are positive and $5^2 + 12^2 = 13^2$ is true. Therefore, (5, 12, 13) is a *Pythagorean triple*.

— Label the statements in the following arguments as p, q, and r, and identify the parts of the argument.

$$\begin{array}{r} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$$

5. If a number is even, then it can be expressed in the form $2n$, where n is an integer. A number of the form $2n$ has a square, $4n^2$, of the form $2(2n^2)$. Therefore, if a number is even, then its square is even.

6. If a number is odd, then it can be expressed in the form $2n + 1$, where n is an integer. A number of the form $2n + 1$ has a square of the form $2(2n^2 + 2n) + 1$. Thus, it follows that if a number is odd, then its square must be odd.

— Study each of the following and determine which, if any, correctly uses the rule of substitution for variables.

7. $\forall x \in \mathrm{R}\quad x + 4 = 4 + x$
$3, 2 \in \mathrm{R}$
$\therefore\ 3 + 4 = 4 + 2$

8. $\forall x \in \mathrm{R}\quad x + 6 = 6 + x$
$6 \in \mathrm{R}$
$\therefore\ 2 + 6 = 6 + 2$

9. $\exists x \in \mathrm{R}\quad x + 4 = 7$
$2 \in \mathrm{R}$
$\therefore\ 2 + 4 = 7$

10. Every rational number can be expressed as the ratio of two integers. The number $0.14\overline{14}$ is a rational number. $\therefore$ $0.14\overline{14}$ can be expressed as the ratio of two integers.

11. All girls are beautiful.
Mary is a girl.
$\therefore$ Mary is beautiful.

12. $\exists x \in \mathrm{R}\quad \frac{x}{6} = 6 \cdot x$
$0 \in \mathrm{R}$
$\therefore\ \frac{0}{6} = 6 \cdot 0$

13. $\forall x \in \mathrm{R}\quad 5x = x \cdot 5$
$8 \in \mathrm{R}$
$\therefore\ 5 \cdot 8 = 8 \cdot 5$

14. All numbers that cannot be expressed as the ratio of two integers are irrational numbers. $\sqrt{2}$ cannot be expressed as the ratio of two integers. $\therefore$ $\sqrt{2}$ is an irrational number.

— Use a truth table to determine whether the conclusion in each of the following is true or false.

15. *Given:* If I am a careful thinker, then I will make an A on tomorrow's logic test.
If I am not a careful thinker, then I will have to study harder.
Conclusion: If I do not make an A on tomorrow's logic test, then I will have to study harder.

16. *Given:* $p \to q$
$\sim p \to r$ (*Hint:* Use the contrapositive of $p \to q$.)
Conclusion: $\sim q \to r$

12–9 Direct Proofs

In the last section you learned some general rules, or patterns, of inference that you can use to prove theorems: *modus ponens*, transitivity, substitution for variables. You will frequently use these rules of inference without explicitly mentioning them by name.

A typical proof may look like this.

Hypothesis: p
Conclusion: s

Statements	*Reasons*
1. $p \to q$	**1.**
2. $q \to r$	**2.**
3. $r \to s$	**3.**
4. $\therefore\ p \to s$	**4.**

The *Hypothesis* is the statement(s) that is (are) understood to be true. The *Conclusion* is the statement that is to be proved true.

The proof then consists in establishing the chain of reasoning under the column labeled *Statements*. Each statement on the left must be justified by a reason on the right. The reason may be that the statement is **Given** in the hypothesis or it may be a **definition, postulate,** or previously proved **theorem.**

In order to make your work with proofs easier, the following review list of postulates and theorems is provided as a reference; they hold for all $x, y, z \in \mathrm{R}$, with division by zero excluded.

Postulate		
2–2	$x + y$ is one and only one real number.	Closure for addition.
2–3	$x + y = y + x$	Commutativity for addition.
2–4	$(x + y) + z = x + (y + z)$	Associativity for addition.
2–5	$x + 0 = 0 + x = x$	Identity for addition.
2–6	$\exists\, -x \in \mathrm{R}$ such that $x + (-x) = 0$.	Inverse for addition.
2–7	xy is one and only one real number.	Closure for multiplication.
2–8	$xy = yx$	Commutativity for multiplication.
2–9	$(xy)z = x(yz)$	Associativity for multiplication.
2–10	$x \cdot 1 = 1 \cdot x = x$	Identity for multiplication.
2–11	$\exists\, \frac{1}{x}$ such that $x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1$	Inverse for multiplication.
2–12	$x(y + z) = xy + xz$	Distributivity.
3–1	If $x = y$ is true, then $x + z = y + z$ is true.	Addition postulate for equations.
3–2	If $x = y$ is true and $z \neq 0$, then $xz = yz$ is true.	Multiplication postulate for equations.

Theorem	
2–3	$x - y = x + (-y)$
2–4	$x \cdot 0 = 0 \cdot x = 0$

The two principal kinds of proof are the direct proof and the indirect proof. In a **direct proof,** you generally begin with a given fact and proceed directly to the desired conclusion by applying the rules of inference previously discussed.

The following examples illustrate this kind of proof.

EXAMPLE 1. Prove Theorem **T–1.**

T–1 $\forall x \in \mathrm{R}\ \forall y \in \mathrm{R}\ \forall z \in \mathrm{R}\quad (x + y)z = xz + yz$

Hypothesis: x, y, and z are arbitrary real numbers.
Conclusion: $(x + y)z = xz + yz$

Statements	*Reasons*
1. x, y, and z are arbitrary real numbers.	1. Given.
2. $(x + y)z = z(x + y)$	2. Postulate 2–8.
3. $z(x + y) = zx + zy$	3. Postulate 2–12.
4. $zx + zy = xz + yz$	4. Why?
5. $(x + y)z = xz + yz$	5. Transitivity.

EXAMPLE 2. Prove Theorem **T–2.**

T–2 The sum of two odd numbers is an even number.

To prove the theorem, you need the following definitions.

Definition 1 An even number is a number of the form $2n$, where n is an integer.

Definition 2 An odd number is a number of the form $2n + 1$, where n is an integer.

Definition 3 If n is an integer, then n is either even or odd.

Hypothesis: a and b are arbitrary odd numbers.
Conclusion: $a + b$ is an even number.

Statements	*Reasons*
1. a and b are odd numbers.	1. Given.
2. Let $a = 2k + 1$, $b = 2p + 1$, and $a + b = 2k + 1 + 2p + 1$.	2. Definition 2.
3. $2k + 1 + 2p + 1 = 2k + 2p + 2$	3. Addition.
4. $2k + 2p + 2 = 2(k + p + 1)$	4. Why?
5. $k + p + 1$ is an integer.	5. Postulate 2–2.
6. $\therefore 2(k + p + 1)$ is an even number.	6. Definition 1.
7. $a + b$ is an even number.	7. Transitivity.

Exercises

A — Use the two examples as a guide and prove the following. (These proved statements may be used in later proofs.)

1. T–3 The square of an even number is even.
2. T–4 The square of an odd number is odd.
3. T–5 The sum of two even numbers is an even number. (*Hint:* Let $2n$ be one even number and let $2k$ be the other. Then find $2n + 2k$.)
4. T–6 The sum of an even number and an odd number is an odd number.
5. T–7 The product of two odd numbers is an odd number.
6. T–8 The product of two even numbers is an even number.
7. T–9 The product of an even number and an odd number is an even number.
8. T–10 The sum of two multiples of 3 is a multiple of 3. (*Hint:* A multiple of 3 is a number of the form $3n$, where n is an integer.)
9. T–11 The product of two multiples of 3 is a multiple of 3.
10. T–12 The sum of two multiples of 6 is a multiple of 6.
11. T–13 If a number is divisible by 6, then it is divisible by 2. (*Hint:* Show that a number divisible by 6 is a multiple of 2.)
12. T–14 If a number is divisible by 6, then it is divisible by 3.
13. T–15 The product of two multiples of 5 is a multiple of 5.
14. T–16 If a number is even, the cube of the number is even.

B — Prove the following.

15. T–17 $\forall a \in R\ \forall b \in R$, the values of the expressions $a - b$ and $b - a$ are opposites. (*Hint:* Prove that $\forall a \in R\ \forall b \in R\ (a - b) + (b - a) = 0$ is true. Then refer to the meaning of opposites.)
16. T–18 If a number is odd, then the cube of the number is odd. (*Hint:* $(2n + 1)^3 = (2n + 1)^2(2n + 1)$)
17. T–19 If one of two given numbers is even and the other is odd, then the square of their sum is an odd number.
18. T–20 If one of two given numbers is even and the other is odd, then the sum of the square of their sum and the odd number is an even number. (*Hint:* Let x be the even number and y the odd number. Prove that $(x + y)^2 + y$ is an even number.)
19. T–21 If a number is not a multiple of 4, then its square is not a multiple of 8. (*Hint:* Numbers that are not multiples of 4 have the form $4n + 1$, $4n + 2$, $4n + 3$, where n is an integer.)
20. T–22 If a is $2uv$, b is $u^2 - v^2$, and c is $u^2 + v^2$, where u and v are positive integers with $u > v$, then $c^2 = a^2 + b^2$ is true. (*Hint:* Examine $(2uv)^2$, $(u^2 - v^2)^2$, and $(u^2 + v^2)^2$.)

12–10 Indirect Proofs

The second principal method of proof is **indirect proof.** The reasoning pattern in an indirect proof is essentially the following.

First, you assume that the statement to be proven is false. Then using this assumption and other given information, you reason from one statement to another until you reach a contradiction. Upon reaching the contradiction, you conclude that it is false to assume that the original statement is false because it leads to a contradiction. Therefore, the original statement must be true.

EXAMPLE 1. Prove the following.

T–23 If the square of an integer, k, is even, then k is even.

Hypothesis: The square of an integer, k, is even.
Conclusion: k is even.

Statements	*Reasons*
1. Assume k is not even.	1. Assumption.
2. If k is not even, then it is odd.	2. An integer that is not even must be odd since $2n \neq 2n+1$ for any integer n (Definitions 1 and 2).
3. If k is odd, then k^2 is odd.	3. Theorem T–4.
4. This is a contradiction, since the square of k is even.	4. Why?
5. k is either even or odd.	5. Definition 3.
6. ∴ It is false that k is odd, so k is not odd; that is, *k is even.*	6. If a statement is false, then its negation is true.

EXAMPLE 2. Prove the following.

T–24 The number $\sqrt{2}$ is an irrational number.

Hypothesis: Given the number $\sqrt{2}$.
Conclusion: $\sqrt{2}$ is an irrational number.

Statements	*Reasons*
1. Assume that $\sqrt{2}$ is not irrational, that is, that it is rational.	1. Assumption.
2. Then $\sqrt{2}$ can be expressed as $\frac{a}{b}$, $b \neq 0$, with a and b integers with no common factors other than 1.	2. Definition of rational number.
3. $\frac{a}{b} = \sqrt{2} \rightarrow a = \sqrt{2}b$	3. Why?
4. $a = \sqrt{2}b \rightarrow a^2 = 2b^2$	4. Why?
5. $a^2 = 2b^2 \rightarrow a^2$ is an even number.	5. Definition 1.
6. If a^2 is even, then a is even.	6. Theorem T–23.
7. If a is even, then it can be expressed as $2n$, where n is an integer.	7. Definition 1.
8. $(2n)^2 = 2b^2$, or $4n^2 = 2b^2$	8. Rule of substitution.
9. $4n^2 = 2b^2 \rightarrow 2n^2 = b^2$	9. Why?
10. If b^2 is $2n^2$, then b is an even number and can be expressed as $2p$.	10. Definition 1.
11. Therefore, $\frac{a}{b}$ is $\frac{2n}{2p}$.	11. Why?
12. But $\frac{a}{b} = \frac{2n}{2p}$ is a contradiction.	12. Why?
13. Thus, it is false that $\sqrt{2}$ is a rational number, so it must be an irrational number.	13. If a statement is false, then its negation is true.

Exercises

A Prove the following indirectly.

1. The number $2\sqrt{2}$ is an irrational number.

2. The number $\frac{1}{4}\sqrt{2}$ is an irrational number.

B **3.** If the square of a number is a multiple of 3, then the number is a multiple of 3. (*Hint:* Assume that the number is not a multiple of 3. Let $3n + 1$ and $3n + 2$ represent numbers that are not multiples of 3. Then examine $(3n + 1)^2$ and $(3n + 2)^2$ and reason to a contradiction.)

4. The number $\sqrt{3}$ is an irrational number.

5. The number $3\sqrt{3}$ is an irrational number.

6. The number $\frac{1}{4}\sqrt{3}$ is an irrational number.

C **7.** If the square of a number is a multiple of 5, then the number is a multiple of 5. (*Hint:* Assume that the number is not a multiple of 5. Let the numbers have the forms $5n + 1$, $5n + 2$, $5n + 3$, $5n + 4$. Then examine the squares of these quantities to reach a contradiction.)

8. The number $\sqrt{5}$ is an irrational number.

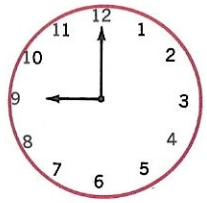
9. If a and b are real numbers and $ab = 0$ is true, then $a = 0$ *or* $b = 0$ is true. (*Hint:* Assume that both a and b are not zero. Then each has a multiplicative inverse. Use this fact with the hypothesis, and reason to a contradiction.)

10. If a is a nonzero real number, then a has exactly one multiplicative inverse. (*Hint:* Assume that a has two distinct multiplicative inverses and reason to a contradiction.)

12–11 Finite Number Fields

So far, you have been proving properties about infinite sets of numbers: the set of even numbers, the set of odd numbers, the set of real numbers. In this section you will examine properties of finite sets of numbers.

An example of a finite set of numbers is the set of numbers on a 12-hour clock.



Suppose you start working at 9 o'clock in the morning and work for 5 hours. You know that you would stop working at 2 o'clock in the afternoon. But suppose you came from some place that did not use a twelve-hour system of time. Then to find out the time you would stop working, you would have to learn how to add numbers on a clock. Hidden in the table on the following page is the pattern. The numbers across the top and down the left are the addends. The numbers in the table

are the sums. Can you uncover the pattern for finding the sums? Cover part of the table and see if you can give the correct sums.

+	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	1
2	3	4	5	6	7	8	9	10	11	12	1	2
3	4	5	6	7	8	9	10	11	12	1	2	3
4	5	6	7	8	9	10	11	12	1	2	3	4
5	6	7	8	9	10	11	12	1	2	3	4	5
6	7	8	9	10	11	12	1	2	3	4	5	6
7	8	9	10	11	12	1	2	3	4	5	6	7
8	9	10	11	12	1	2	3	4	5	6	7	8
9	10	11	12	1	2	3	4	5	6	7	8	9
10	11	12	1	2	3	4	5	6	7	8	9	10
11	12	1	2	3	4	5	6	7	8	9	10	11
12	1	2	3	4	5	6	7	8	9	10	11	12

You should see that the numbers are continuous through 12 and then start over from 1. If the real-number sum exceeds 12, the clock sum can be quickly found by subtracting 12 from the real-number sum. The real-number sum of 10 and 9 is 19; the clock sum is 19 − 12, or 7.

The following table illustrates clock multiplication. Cover part of the table and see if you can give the correct products.

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	2	4	6	8	10	12
3	3	6	9	12	3	6	9	12	3	6	9	12
4	4	8	12	4	8	12	4	8	12	4	8	12
5	5	10	3	8	1	6	11	4	9	2	7	12
6	6	12	6	12	6	12	6	12	6	12	6	12
7	7	2	9	4	11	6	1	8	3	10	5	12
8	8	4	12	8	4	12	8	4	12	8	4	12
9	9	6	3	12	9	6	3	12	9	6	3	12
10	10	8	6	4	2	12	10	8	6	4	2	12
11	11	10	9	8	7	6	5	4	3	2	1	12
12	12	12	12	12	12	12	12	12	12	12	12	12

Does the set of clock numbers possess any of the properties of the set of real numbers? For example, are the clock numbers closed under addition and multiplication? Are the clock numbers commutative for addition and multiplication? Do they have identity elements? Do they have inverse elements?

A set of numbers that satisfies Postulates 2–2 through 2–12 (see page 528) is called a **field.**

Definition The set F forms a field with respect to two operations $\oplus$ and $\otimes$ if and only if
i) the set is closed, commutative, and associative under the two operations,
ii) there is an identity element z for $\oplus$ and an identity element e for $\otimes$,
iii) each element has an inverse element under $\oplus$, and each element except z has an inverse element under $\otimes$, and
iv) $\otimes$ is distributive with respect to $\oplus$.

The set of real numbers is an example of a field. Is the set of clock numbers, $C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, a field? The following discussion will answer the question.

Postulate	Does set C satisfy it?
1. $a + b$ is an element in the set. 2. ab is an element in the set.	1.–2. Yes. The sum or product of any two numbers is in C because every answer in the table is a clock number.
3. $a + b = b + a$ 4. $ab = ba$	3.–4. Yes. Check a few cases. It is easily seen by drawing diagonals on the tables from upper left to lower right. It divides the numbers symmetrically: the pattern on one side is exactly the same as on the other.
5. $(a + b) + c = a + (b + c)$ 6. $(ab)c = a(bc)$	5.–6. Yes. Many cases are required to show that clock numbers do satisfy this. Choose a few examples and try it.
7. There is an identity element for addition.	7. Yes. 12 added to any number gives that number.
8. There is an identity element for multiplication.	8. Yes. 1 times any number gives that number.

(Continued on next page.)

9. Each element has an additive inverse.

9. Yes. The additive identity element, 12, appears in each column. So each number has an additive inverse.

10. Each element has a multiplicative inverse.

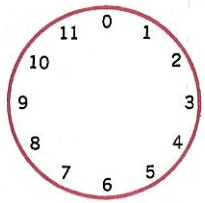
10. No. There is no number by which to multiply 2, 3, 4, 6, 8, 9, 10, or 12 to get the identity element, 1; so they do not have multiplicative inverses.

11. $a(b + c) = ab + ac$

11. Yes. All combinations would have to be checked to prove this. Try a few to verify it.

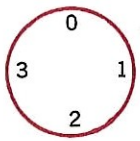
If one postulate is not satisfied, then the set of numbers is not a field. Thus, you could have stopped after finding that postulate 10 did not hold. The set of clock numbers is not a field.

Note that in the finite set C under the operations of clock arithmetic, 12 acts the same as 0 does in the set of real numbers; that is, 12 is the additive identity element and 12 times any number is 12. Therefore, if the clock had been labeled as shown at the right, the pattern for addition and multiplication would be the same with 12 being replaced by 0.



Exercises

A **1.** Complete the following addition and multiplication tables for the set of numbers on a "four-hour clock": {0, 1, 2, 3}.

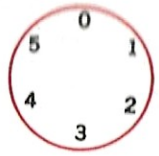


+	0	1	2	3
0	0	1	2	3
1	1	?	?	?
2	2	3	?	?
3	3	?	?	2

×	0	1	2	3
0	0	0	0	0
1	0	?	?	?
2	0	2	?	?
3	0	?	?	1

2. Does the set of numbers on a "four-hour clock" meet all the requirements to be a field? If not, which postulate(s) is (are) not satisfied? Give examples.

3. Complete the following tables for a "six-hour clock."



+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	?	?	?	?	0
2	2	?	4	?	?	?
3	3	?	?	?	?	?
4	4	?	?	?	2	?
5	5	?	1	?	?	?

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	?	2	?	?	?
2	0	?	?	?	?	?
3	0	?	?	?	0	?
4	0	4	?	?	?	2
5	0	?	4	?	?	1

4. Does the set of numbers on a "six-hour clock" meet all the requirements to be a field? Explain.

5. Is any set of numbers on an "even-number clock" a field? Explain.

6. Complete the following tables for a "three-hour clock."

+	0	1	2
0	0	?	?
1	?	2	?
2	?	?	1

×	0	1	2
0	0	0	0
1	0	?	?
2	0	?	?

7. Is the set of numbers on a "three-hour clock" a field?

8. Is the set of numbers on a "five-hour clock" a field?

9. Is the set of numbers on a "nine-hour clock" a field?

10. Is any set of numbers on an "odd-number clock" a field? Explain.

B **11.** Is there any pattern that indicates which sets of "clock numbers" are fields? If so, explain.

12. The following is a finite number system that has two numbers, 0 and 1. The addition and multiplication tables for this system are given below.

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

Does this finite number system meet the requirements to be a field? Explain.

13. The following finite number system has elements p, a, b, and c, instead of numbers. The operation $*$ for this system is unlike either ordinary addition or multiplication.

$*$	p	a	b	c
p	p	a	b	c
a	a	p	c	b
b	b	c	p	a
c	c	b	a	p

a. Has this system an identity element? If so, what is it?

b. Does each element have an inverse element? If so, what is it for each element?

c. Is the system commutative?

d. Is the system associative?

e. Is the system a field? Explain.

CHAPTER REVIEW

1. Know the meaning of and be able to use each of the following words or phrases. The number shown after each word or phrase indicates where it is introduced, in case you need to review.

statement (*504*)
simple statement (*505*)
compound statement (*505*)
conjunction (*505*)
truth value (*506*)
truth table (*506*)
disjunction (*506*)
conditional statement (*509*)
converse (*510*)
biconditional (*511*)
negation (*513*)
inverse (*513*)
logically equivalent statements (*514*)
tautology (*516*)
Law of the Excluded Middle (*517*)
quantifiers (*518*)
premise, or hypothesis (*523*)
conclusion (*523*)
inference (*523*)
modus ponens (*523*)
transitivity (*524*)
rule of substitution for variables (*525*)
direct proof (*528*)
indirect proof (*531*)
field (*535*)

2. Why is the sentence $3x - 2 = 7$ *not* a statement?

3. Why *is* $\forall x \in \mathbb{R} \quad 3(x + y) = 3x + 3y$ a statement?

4. Explain the difference between a compound statement that is a conjunction and one that is a disjunction.

5. Explain why a true conjunction can be represented by the intersection of sets and a true disjunction by the union of sets.

6. Complete this table by inserting the correct truth values.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
T	T	?	?	?
T	F	?	?	?
F	T	?	?	?
F	F	?	?	?

7. Write the inverse of this statement: *If a number may be represented by the ratio of two integers, it is a rational number.*
Is the inverse true? Explain.

8. If you are given that p is true and q is false, then $\sim p$ is false and $\sim q$ is true. Explain why $\sim p \wedge \sim q$ is false.

9. Give an example of a tautology.

10. *All whole numbers are rational numbers.* Is this a true statement? Write the negation of the statement. Is it true?

11. *For any given acute angle, the ratio of the measure of the side opposite the angle to the measure of the side adjacent is called the tangent.* Why is this not a good definition?

12. Represent the following statements symbolically, and explain how the inference is an example of the rule of substitution for variables.

Drivers of sports cars pay higher insurance premiums.
Sam drives a sports car.
$\therefore$ Sam pays a higher insurance premium.

13. Prove: If a number is odd, then the cube of the number is odd.

14. What is the first step in an indirect proof? How do you then establish the conclusion?

15. Is the set of integers a field under addition and multiplication? Explain.

CHAPTER TEST

1. Which of the properties of a good definition does the following definition fail to have? *A rational number is a fraction.*

2. Find the negation of the following statements.

a. π is an irrational number.

b. Some triangles are isosceles.

3. Complete the following truth tables.

Conjunction

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

Disjunction

p	q	$p \vee q$
T	T	
T	F	
F	T	
F	F	

Conditional

p	q	$p \rightarrow q$
T	T	
T	F	
F	T	
F	F	

Biconditional

p	q	$p \leftrightarrow q$
T	T	
T	F	
F	T	
F	F	

4. Given the conditional $p \rightarrow q$, form its converse.

5. Use a truth table to determine whether or not the following statement is a tautology. $(p \vee q) \rightarrow (p \wedge q)$

6. Indicate the method of proof used in reaching each conclusion.

a. $\begin{array}{l} p \rightarrow q \\ \underline{p} \\ \therefore q \end{array}$ b. $\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$ c. $\begin{array}{l} \begin{cases} \forall x \in \mathrm{R} & x + x = 2x \\ 3 \in \mathrm{R} \end{cases} \\ \hline \therefore 3 + 3 = 2 \cdot 3 \end{array}$

Prove the following.

7. $\forall x \in \mathrm{R} \quad 5(x + 2) = 5x + 10$

8. The sum of three odd numbers is an odd number.

9. If a number is divisible by 2, its square is divisible by 2.

10. $\sqrt{2}$ is irrational.

11. What is the general method used in an indirect proof?

12. Is the set of whole numbers a field under addition and multiplication? Explain your answer.

CUMULATIVE REVIEW

Give the letter of the response that best completes the sentence or answers the question.

1. Given that W is the set of whole numbers, $\underline{?}$ is true.

a. $\frac{5}{6} \in \mathrm{W}$ b. $1\frac{1}{2} \in \mathrm{W}$ c. $\sqrt{2} \in \mathrm{W}$ d. $3 \in \mathrm{W}$

2. If the replacement set for x and y is the set of real numbers, which of the following is true?

a. $\forall x \, \forall y \quad x + 1 = x$ b. $\forall x \, \forall y \quad x \cdot y = x + y$
c. $\forall x \, \forall y \quad (x + y) + x = x + (y + x)$ d. $\forall x \, \forall y \quad x \cdot y > x \cdot x$

3. Which of the following pairs are equivalent?

a. $10 - (-7)$ and $10 + (-7)$
b. $-10 - (-7)$ and $10 - (-7)$
c. $10 - (-7)$ and $10 + (7)$
d. $-10 + (-7)$ and $-10 - (-7)$

4. What is the solution set of the equation $|2x - 3| + 8 = 9$?

a. $\{-2, -1\}$ b. $\{2, -1\}$ c. $\{2, 1\}$ d. $\{-2, 1\}$

5. Which of the following expressions is equivalent to $\frac{2}{3}(a + b) + \frac{3}{4}(2a - b)$?

a. $\frac{13a}{6} + \frac{b}{12}$ b. $\frac{26a - b}{12}$
c. $\frac{13a - b}{12}$ d. $\frac{26a + b}{12}$

6. What is the graph of the solution set of $3x - 2 > 15$?

a. A half-plane. b. The coordinate plane.
c. A line segment. d. A ray.

7. Which of the following relations is not a function?

a. $y = \pm\sqrt{x}$ b. $x = \pm\sqrt{y}$
c. $3x - 2 = 5y$ d. $5x = 2y - 7$

8. What is the equation of the line parallel to the x axis that passes through the intersection of the graph of this system?

$$\begin{cases} -7x + 2y = 10 \\ 5x - 6y = 2 \end{cases}$$

a. $y = 2$ b. $y = -2$ c. $x = 2$ d. $x = -2$

9. What two consecutive integers may replace x and y to make $x < -\sqrt{101} < y$ true?

a. 11 and 10 b. 10 and 11
c. -11 and -10 d. -10 and -11

10. Which of the following systems of equations is independent?

a. $\begin{cases} 3x - 2y = 7 \\ x + 5y = 1 \end{cases}$ b. $\begin{cases} 2x - y = 4 \\ 4x - 2y = 8 \end{cases}$
c. $\begin{cases} 3x - y = 5 \\ 9x - 3y = 2 \end{cases}$ d. $\begin{cases} x + 3y = 2 \\ 5x + 15y = -7 \end{cases}$

a. x^4y^6 **b.** x^4y^{-4} **c.** $\frac{x^6}{y^4}$ **d.** x^6y^4

12. Consider the trinomial $ax^2 + bx + c$. If a and c are each perfect squares, then which is the correct value of b to make the trinomial a perfect square?

a. $2\sqrt{a} \cdot \sqrt{c}$ **b.** $(2\sqrt{a})(2\sqrt{c})$

c. $\frac{\sqrt{a} \cdot \sqrt{c}}{2}$ **d.** $\sqrt{a} \cdot \sqrt{c}$

13. The graph of $y = 3x^2 + 5x - 6$ intersects the x axis

a. at just the origin. **b.** in two points.

c. in one point that is not the origin. **d.** in no points.

14. What are the solutions of the equation $15x^2 + 2\sqrt{2}x - 2 = 0$?

a. $-\frac{\sqrt{2}}{5}$ and $\frac{\sqrt{2}}{3}$ **b.** $\frac{5}{\sqrt{2}}$ and $\frac{3}{\sqrt{2}}$

c. $\frac{\sqrt{2}}{5}$ and $-\frac{\sqrt{2}}{3}$ **d.** $-\frac{\sqrt{2}}{5}$ and $-\frac{\sqrt{2}}{3}$

15. Simplifying $\frac{\frac{3x^2 - 3y^2}{6(x+y)}}{5}$ gives

a. $\frac{18(x^3 + x^2y - xy^2 - y^3)}{5}$. **b.** $\frac{5(x-y)}{2}$.

c. $\frac{5x - y}{2}$. **d.** $\frac{18(x^2 - y^2)}{5}$.

16. If you are given right triangle ABC with $|\angle B| = 90°$, $|\angle C| = 54°$, and $|\overline{AC}| = 38$ feet, which of the following would you use to find $|\overline{AB}|$?

a. The tangent ratio. **b.** The sine ratio.

c. The cosine ratio. **d.** The Pythagorean theorem.

17. How may the logarithmic function $y = \log_b x$ be written as an exponential function?

a. $y = b^x$ **b.** $x^b = y$ **c.** $x = b^y$ **d.** $y^b = x$

18. A contractor needs concrete for a construction project. If a large machine can mix the required amount in 12 hours and a small one can mix it in 20 hours, how long will it take both machines running together to mix the required amount of concrete?

a. 15 hours **b.** 7.5 hours

c. 10.3 hours **d.** 17.5 hours

19. If y varies inversely as x, and y is 5 when x is 3, what is the value of y when x is 6?

a. 2.5 **b.** 10 **c.** $3\frac{3}{5}$ **d.** 1.25

20. If you are given the statement $\sim p \rightarrow q$, then ? is its inverse.

a. $\sim p \wedge q$ **b.** $\sim p \rightarrow \sim q$

c. $p \rightarrow q$ **d.** $p \rightarrow \sim q$

21. Let the domain of a relation be the automobile license plates in a certain state, and let the range be the automobiles owned by people in that same state. Then mapping of license plates onto automobiles is an example of a

a. one-to-many mapping. **b.** one-to-one mapping.

c. many-to-one mapping. **d.** linear function.

22. Which of the following is equivalent to $\sim(p \vee q)$?

a. $\sim p \wedge \sim q$ **b.** $\sim p \vee \sim q$

c. $\sim p \rightarrow \sim q$ **d.** $p \wedge \sim q$

ENRICHMENT

Faulty Logic

If you were asked to solve the equation $3x + 5 - 2x = 3x$, what would you do? Almost certainly you would start applying postulates to collect the x terms on one side of the equals sign and the constants on the other side. Then you would divide by the coefficient of the x terms and arrive at an answer. In the given equation, you get $5 = 2x$, or $x = \frac{5}{2}$. Some of you may check $x = \frac{5}{2}$ in the original equation, but others may feel that the check is unnecessary if you have worked carefully. The goal of this section is to show logically that the check is the most important part of the procedure.

When you are asked to solve the equation $3x + 5 - 2x = 3x$, you are being asked to find a value of x, call it x_1, which makes the equation true; that is, you are to supply some number for x_1. But you do not do this with the usual method of solution. You automatically assume that there is an x such that $3x + 5 - 2x = 3x$ is known to be a true equality when $x = x_1$. In short you are proving that "If $3x + 5 - 2x = 3x$ does have a solution x_1, then x_1 is $\frac{5}{2}$." It is the check that really shows that "If x_1 is $\frac{5}{2}$, then $3x + 5 - 2x = 3x$ does have a solution x_1," which is the converse of the other statement.

braic equation you started with has a solution. Otherwise, some very strange results occur. For example, $\frac{1}{x}+\frac{2}{x+1}=\frac{3}{x}$ looks innocent enough, but examine the following solution.

$$\frac{1}{x} + \frac{2}{x+1} = \frac{3}{x}$$

$$\frac{x(x+1)1}{x} + \frac{x(x+1)2}{x+1} = \frac{x(x+1)3}{x}$$

$$(x+1) + 2x = 3x+3$$

$$3x + 1 = 3x+3$$

$$1 = 3$$

The result is disturbing, and no amount of checking will tell you what is wrong. What actually has happened is that you have done an indirect "disproof." You assumed that $\frac{1}{x}+\frac{2}{x+1}=\frac{3}{x}$ was a valid equality and derived a contradiction. Therefore, your assumption is invalid, and the equation is not a valid equation.

Here is a slightly different example.

$$\frac{x}{x+2} - \frac{4}{x+1} = \frac{-2}{x+2}$$

$$x(x+1) - 4(x+2) = -2(x+1)$$

$$x^2 - x - 6 = 0$$

$$(x=-2) \vee (x=3)$$

Again checking reveals trouble, for $x=3$ is a solution, but $x=-2$ is not. It could have been seen ahead of time that $x=-1$ and $x=-2$ would give zero denominators, but the exception was not made.

Here is a final example.

Since $\sqrt{ab}=\sqrt{a}\sqrt{b}$ is true, you may write $\sqrt{(-1)(-1)}=\sqrt{-1}\sqrt{-1}$. But you know that $\sqrt{(-1)(-1)}=\sqrt{1}=1$, and by the very definition of square root $\sqrt{-1}\cdot\sqrt{-1}=-1$. Therefore, $1=-1$ must be true. What went wrong? Check back in Chapter 6 to see if you may assume $\sqrt{ab}=\sqrt{a}\sqrt{b}$ for *any* a and b.

Exercises

1. Solve the following equations and explain the results.

a. $\frac{6}{x+2}+\frac{7}{x}=\frac{13}{x+2}$

b. $\frac{7}{x-1}-\frac{6}{x^2-1}=5$

c. $\frac{x^2+5}{x-3}=x+3$

d. $\frac{2x}{x+2}-\frac{2}{x-2}=\frac{-4}{x+2}$

2. Let x and y be any two real numbers and let $x+y=2z$. Multiply both sides by $x-y$ to get $x^2-y^2=2zx-2zy$. Now add $z^2-z^2=2xz$ to both sides to get $x^2-2xz+z^2=y^2-2zy+z^2$, or $(x-z)^2=(y-z)^2$. Then $\sqrt{(x-z)^2}=\sqrt{(y-z)^2}$, or $x-z=y-z$, or $x=y$. Thus, since x and y were any real numbers, any two real numbers are equal. What is wrong? (*Hint:* $\sqrt{5^2}$ and $\sqrt{(-5)^2}$ both equal $\sqrt{25}=5$ since $\sqrt{a}$ stands for positive square root of $+a$.)

3. The following "proof" assumes that certain laws of exponents hold in certain cases. Explain the erroneous steps.

$$\sqrt{-1}=\sqrt{-1}$$

$$\sqrt{\frac{1}{-1}}=\sqrt{\frac{-1}{1}}$$

$$\frac{\sqrt{1}}{\sqrt{-1}}=\frac{\sqrt{-1}}{\sqrt{1}} \rightarrow \sqrt{1}\sqrt{1}=\sqrt{-1}\sqrt{-1} \rightarrow 1=-1$$

4. Let $\frac{a}{b}=\frac{c}{d}$ and $a=c$. Then $ad=bc$, and since $a=c$, $ad-bc=0$ or $ad-ba=0$. Then $a(d-b)=0$, so $a=0$ or $d=b$. This shows that if two fractions are equal and have equal nonzero numerators, then the denominators are also equal.

Now consider the equation $\frac{x}{x-2}+2=\frac{3x-4}{x-1}$. If you put the left side over a common denominator, you have $\frac{3x-4}{x-2}=\frac{3x-4}{x-1}$. But then by the little "theorem" above, $x-2=x-1$, or $2=1$. What went wrong?

THE GROWTH OF MATHEMATICS

Vieta

Franciscus Vieta (1540–1603) was a lawyer in France and spent his life in service to the king. He studied mathematics in his leisure time, but he served the king as mathematician, too. Henry IV was challenged to find someone who could solve an equation of the forty-fifth degree that had been solved by a Belgian. Vieta recognized that it could be treated as a trigonometric function and found twenty-three solutions.

Vieta's greatest contribution was that of *inventing* symbolic algebra, although other men made it generally accepted. Whereas Diophantus had used word abbreviations and had only one symbol for the unknown, Vieta used the whole alphabet with numerals and signs. He did not have symbolism for exponents or equality, but he used x *quadratus* and x *cubus* for x^2 and x^3, respectively, rather than different letters as was then the practice. He is truly the father of symbolic algebra.

Table of Squares and Square Roots

No.	Square	Square Root	No.	Square	Square Root	No.	Square	Square Root
1	1	1.000	51	2601	7.141	101	10,201	10.050
2	4	1.414	52	2704	7.211	102	10,404	10.100
3	9	1.732	53	2809	7.280	103	10,609	10.149
4	16	2.000	54	2916	7.348	104	10,816	10.198
5	25	2.236	55	3025	7.416	105	11,025	10.247
6	36	2.449	56	3136	7.483	106	11,236	10,296
7	49	2.646	57	3249	7.550	107	11,449	10.344
8	64	2.828	58	3364	7.616	108	11,664	10.392
9	81	3.000	59	3481	7.681	109	11,881	10.440
10	100	3.162	60	3600	7.746	110	12,100	10.488
11	121	3.317	61	3721	7.810	111	12,321	10.536
12	144	3.464	62	3844	7.874	112	12,544	10.583
13	169	3.606	63	3969	7.937	113	12,769	10.630
14	196	3.742	64	4096	8.000	114	12,996	10.677
15	225	3.873	65	4225	8.062	115	13,225	10.724
16	256	4.000	66	4356	8.124	116	13,456	10.770
17	289	4.123	67	4489	8.185	117	13,689	10.817
18	324	4.243	68	4624	8.246	118	13,924	10.863
19	361	4.359	69	4761	8.307	119	14,161	10.909
20	400	4.472	70	4900	8.367	120	14,400	10.954
21	441	4.583	71	5041	8.426	121	14,641	11.000
22	484	4.690	72	5184	8.485	122	14,884	11.045
23	529	4.796	73	5329	8.544	123	15,129	11.091
24	576	4.899	74	5476	8.602	124	15,376	11.136
25	625	5.000	75	5625	8.660	125	15,625	11.180
26	676	5.099	76	5776	8.718	126	15,876	11.225
27	729	5.196	77	5929	8.775	127	16,129	11.269
28	784	5.292	78	6084	8.832	128	16,384	11.314
29	841	5.385	79	6241	8.888	129	16,641	11.358
30	900	5.477	80	6400	8.944	130	16,900	11.402
31	961	5.568	81	6561	9.000	131	17,161	11.446
32	1024	5.657	82	6724	9.055	132	17,424	11.489
33	1089	5.745	83	6889	9.110	133	17,689	11.533
34	1156	5.831	84	7056	9.165	134	17,956	11.576
35	1225	5.916	85	7225	9.220	135	18,225	11.619
36	1296	6.000	86	7396	9.274	136	18,496	11.662
37	1369	6.083	87	7569	9.327	137	18,769	11.705
38	1444	6.164	88	7744	9.381	138	19,044	11.747
39	1521	6.245	89	7921	9.434	139	19,321	11.790
40	1600	6.325	90	8100	9.487	140	19,600	11.832
41	1681	6.403	91	8281	9.539	141	19,881	11.874
42	1764	6.481	92	8464	9.592	142	20,164	11.916
43	1849	6.557	93	8649	9.644	143	20,449	11.958
44	1936	6.633	94	8836	9.695	144	20,736	12.000
45	2025	6.708	95	9025	9.747	145	21,025	12.042
46	2116	6.782	96	9216	9.798	146	21,316	12.083
47	2209	6.856	97	9409	9.849	147	21,609	12.124
48	2304	6.928	98	9604	9.899	148	21,904	12.166
49	2401	7.000	99	9801	9.950	149	22,201	12.207
50	2500	7.071	100	10,000	10.000	150	22,500	12.247

Table of Sines, Cosines, and Tangents

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
0°	.0000	1.0000	.0000	45°	.7071	.7071	1.0000
1	.0175	.9998	.0175	46	.7193	.6947	1.0355
2	.0349	.9994	.0349	47	.7314	.6820	1.0724
3	.0523	.9986	.0524	48	.7431	.6691	1.1106
4	.0698	.9976	.0699	49	.7547	.6561	1.1504
5	.0872	.9962	.0875	50	.7660	.6428	1.1918
6	.1045	.9945	.1051	51	.7771	.6293	1.2349
7	.1219	.9925	.1228	52	.7880	.6157	1.2799
8	.1392	.9903	.1405	53	.7986	.6018	1.3270
9	.1564	.9877	.1584	54	.8090	.5878	1.3764
10	.1736	.9848	.1763	55	.8192	.5736	1.4281
11	.1908	.9816	.1944	56	.8290	.5592	1.4826
12	.2079	.9781	.2126	57	.8387	.5446	1.5399
13	.2250	.9744	.2309	58	.8480	.5299	1.6003
14	.2419	.9703	.2493	59	.8572	.5150	1.6643
15	.2588	.9659	.2679	60	.8660	.5000	1.7321
16	.2756	.9613	.2867	61	.8746	.4848	1.8040
17	.2924	.9563	.3057	62	.8829	.4695	1.8807
18	.3090	.9511	.3249	63	.8910	.4540	1.9626
19	.3256	.9455	.3443	64	.8988	.4384	2.0503
20	.3420	.9397	.3640	65	.9063	.4226	2.1445
21	.3584	.9336	.3839	66	.9135	.4067	2.2460
22	.3746	.9272	.4040	67	.9205	.3907	2.3559
23	.3907	.9205	.4245	68	.9272	.3746	2.4751
24	.4067	.9135	.4452	69	.9336	.3584	2.6051
25	.4226	.9063	.4663	70	.9397	.3420	2.7475
26	.4384	.8988	.4877	71	.9455	.3256	2.9042
27	.4540	.8910	.5095	72	.9511	.3090	3.0777
28	.4695	.8829	.5317	73	.9563	.2924	3.2709
29	.4848	.8746	.5543	74	.9613	.2756	3.4874
30	.5000	.8660	.5774	75	.9659	.2588	3.7321
31	.5150	.8572	.6009	76	.9703	.2419	4.0108
32	.5299	.8480	.6249	77	.9744	.2250	4.3315
33	.5446	.8387	.6494	78	.9781	.2079	4.7046
34	.5592	.8290	.6745	79	.9816	.1908	5.1446
35	.5736	.8192	.7002	80	.9848	.1736	5.6713
36	.5878	.8090	.7265	81	.9877	.1564	6.3138
37	.6018	.7986	.7536	82	.9903	.1392	7.1154
38	.6157	.7880	.7813	83	.9925	.1219	8.1443
39	.6293	.7771	.8098	84	.9945	.1045	9.5144
40	.6428	.7660	.8391	85	.9962	.0872	11.4301
41	.6561	.7547	.8693	86	.9976	.0698	14.3007
42	.6691	.7431	.9004	87	.9986	.0523	19.0811
43	.6820	.7314	.9325	88	.9994	.0349	28.6363
44	.6947	.7193	.9657	89	.9998	.0175	57.2900
45	.7071	.7071	1.0000	90	1.0000	.0000	

Table of Common Logarithms

N	0	1	2	3	4	5	6	7	8	9
1.0	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
1.1	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
1.2	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
1.3	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
1.4	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
1.5	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
1.6	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
1.7	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
1.8	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
1.9	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
2.0	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
2.1	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
2.2	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
2.3	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
2.4	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
2.5	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
2.6	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
2.7	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
2.8	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
2.9	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
3.0	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
3.1	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
3.2	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
3.3	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
3.4	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
3.5	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
3.6	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
3.7	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
3.8	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
3.9	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
4.0	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
4.1	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
4.2	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
4.3	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
4.4	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
4.5	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
4.6	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
4.7	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
4.8	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
4.9	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
5.0	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
5.1	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
5.2	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
5.3	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
5.4	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

Table of Common Logarithms

N	0	1	2	3	4	5	6	7	8	9
5.5	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
5.6	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
5.7	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
5.8	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
5.9	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
6.0	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
6.1	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
6.2	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
6.3	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
6.4	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
6.5	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
6.6	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
6.7	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
6.8	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
6.9	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
7.0	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
7.1	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
7.2	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
7.3	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
7.4	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
7.5	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
7.6	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
7.7	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
7.8	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
7.9	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
8.0	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
8.1	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
8.2	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
8.3	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
8.4	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
8.5	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
8.6	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
8.7	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
8.8	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
8.9	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
9.0	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
9.1	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
9.2	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
9.3	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
9.4	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
9.5	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
9.6	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
9.7	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
9.8	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
9.9	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

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